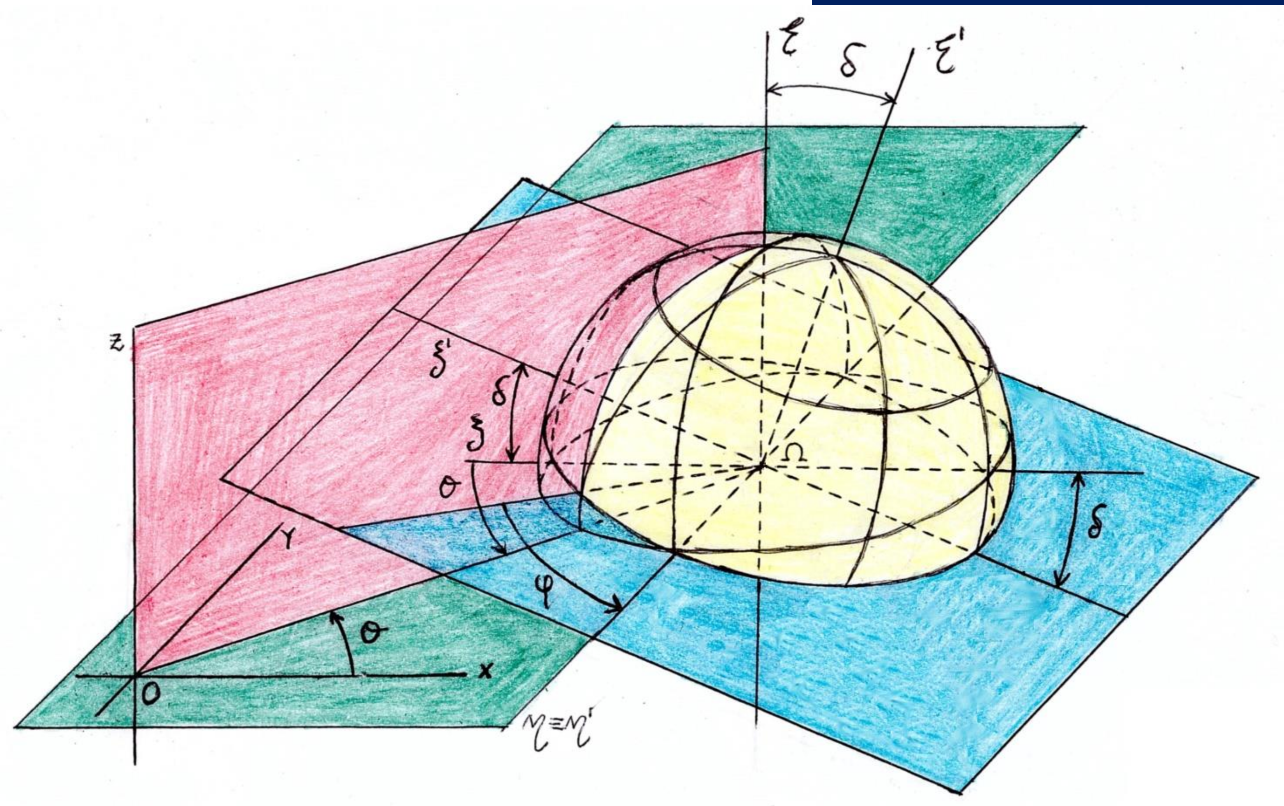
2019

Solar radiation



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1 Spherical trigonometry

1.1 Introduction

This chapter aims to introduce a code that – for any given latitude and any given day of the year – can calculate the inclination of the sunrays and the hours of dawn and dusk. To do that, we must solve a series of problems of spherical trigonometry and give a mathematical definition for dawn and dusk.

1.2 Coordinate systems and angles

Let the origin of the coordinate system $O; x, y, z$ be the centre of the Sun. The green plane (plane *xy*) is the plane of the orbit of the Earth around the Sun. The centre of the Earth is Ω while the coordinate system $\Omega; \xi', \eta', \zeta'$ is integral with planet Earth. The coordinate system $\Omega; \xi, \eta, \zeta$, on the other hand, maintains the same orientation, while Earth rotates around the Sun. The inclination of the axis of the Earth with respect to ζ is given by δ. The latitude is λ. The angle θ defines the day of the year, while the angle $\varphi + \psi$ gives the hour of the day. Given these settings and the orientation of the tilting angle with respect to $O; x, y, z$ in Figure 1, we have:

Eq. 1
$$\begin{cases} \theta = 0 \leftarrow winter\ solstice\ (21/22\ December) \\ \theta = \frac{\pi}{2} \leftarrow spring\ equinox\ (20\ March) \\ \theta = \pi \leftarrow\ summer\ solstice\ (21\ June) \\ \theta = \frac{3\pi}{2} \leftarrow autumn\ equinox\ (22/23\ September) \end{cases}$$

1.3 Unit vector orthogonal to the surface of the Earth

We aim to calculate the angle between sun rays and the unit vector $\hat{n}$, which is orthogonal to the surface of the Earth in point *P*. To accomplish that, we just need to find the expression of $\hat{n}$ with respect to $O; x, y, z$. In $\Omega; \xi', \eta', \zeta'$ the expression of $\hat{n}$ is

$$\hat{n} = -\cos\lambda \sin\psi\, \hat{\varepsilon}_1' + \cos\lambda \cos\psi\, \hat{\varepsilon}_2' + \sin\lambda\, \hat{\varepsilon}_3'$$

Let's now find the relationships between $\hat{\varepsilon}_1', \hat{\varepsilon}_2', \hat{\varepsilon}_3'$ and $\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3$. We have:

Eq. 2
$$\begin{cases} \hat{\varepsilon}_1' = \cos\delta\, \hat{\varepsilon}_1 + \sin\delta\, \hat{\varepsilon}_3 \\ \hat{\varepsilon}_2' = \hat{\varepsilon}_2 \\ \hat{\varepsilon}_3' = -\sin\delta\, \hat{\varepsilon}_1 + \cos\delta\, \hat{\varepsilon}_3 \end{cases} \Rightarrow \begin{bmatrix} \hat{\varepsilon}_1' \\ \hat{\varepsilon}_2' \\ \hat{\varepsilon}_3' \end{bmatrix} = \begin{bmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & \cos\delta \end{bmatrix} \begin{bmatrix} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \\ \hat{\varepsilon}_3 \end{bmatrix}$$

So, the expression of $\hat{n}$ with respect to $\Omega; \xi, \eta, \zeta$ is

$$\hat{n} = -\cos\lambda \sin\psi\, (\cos\delta\, \hat{\varepsilon}_1 + \sin\delta\, \hat{\varepsilon}_3) + \cos\lambda \cos\psi\, \hat{\varepsilon}_2 + \sin\lambda\, (-\sin\delta\, \hat{\varepsilon}_1 + \cos\delta\, \hat{\varepsilon}_3) =$$

$$= -\cos\lambda \sin\psi \cos\delta\, \hat{\varepsilon}_1 - \cos\lambda \sin\psi \sin\delta\, \hat{\varepsilon}_3 + \cos\lambda \cos\psi\, \hat{\varepsilon}_2 - \sin\lambda \sin\delta\, \hat{\varepsilon}_1 + \sin\lambda \cos\delta\, \hat{\varepsilon}_3 \Rightarrow$$

Eq. 3 $$\hat{n} = -(\cos\lambda \sin\psi \cos\delta + \sin\lambda \sin\delta)\hat{\varepsilon}_1 + \cos\lambda \cos\psi\, \hat{\varepsilon}_2 + (\sin\lambda \cos\delta - \cos\lambda \sin\psi \sin\delta)\hat{\varepsilon}_3$$

Now, how can we express $\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3$ in function of $\hat{e}_1, \hat{e}_2, \hat{e}_3$?

Eq. 4 $$\begin{cases} \hat{\varepsilon}_1 = -\hat{e}_1 \\ \hat{\varepsilon}_2 = -\hat{e}_2 \\ \hat{\varepsilon}_3 = \hat{e}_3 \end{cases} \Rightarrow \begin{bmatrix} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \\ \hat{\varepsilon}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

This means that the expression of $\hat{n}$ with respect to $O; x, y, z$ is

Eq. 5 $$\hat{n} = (\cos\lambda \sin\psi \cos\delta + \sin\lambda \sin\delta)\hat{e}_1 - \cos\lambda \cos\psi\, \hat{e}_2 + (\sin\lambda \cos\delta - \cos\lambda \sin\psi \sin\delta)\hat{e}_3$$

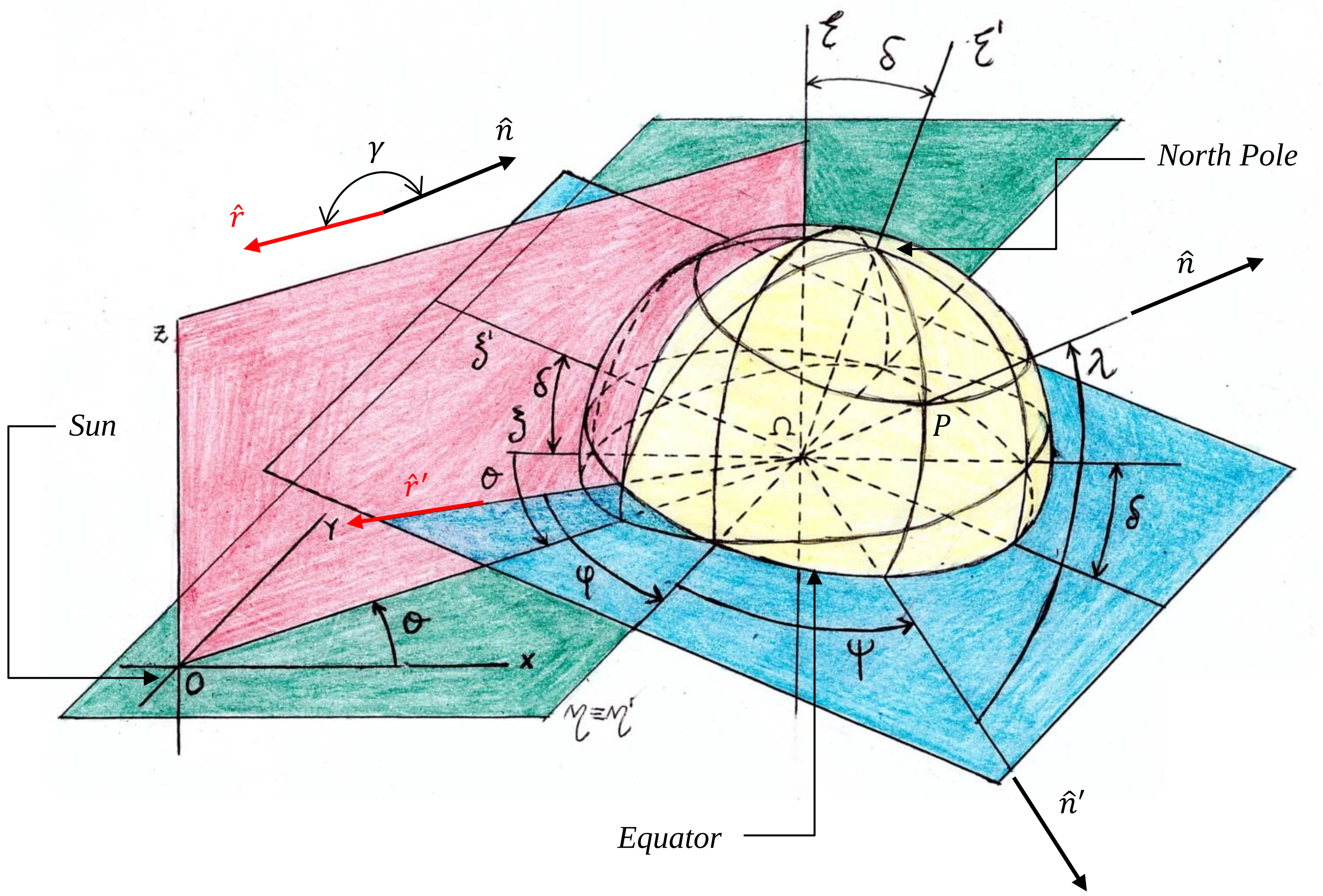


Figure 1. Coordinate systems and angular coordinates used in this paper.

1.4 The inclination of Sun rays with respect to $\hat{n}$

The angle γ between sun rays and $\hat{n}$ can be calculated considering the unit vector

Eq. 6 $$\hat{r} = -\cos\theta\, \hat{e}_1 - \sin\theta\, \hat{e}_2$$

We have

Eq. 7 $$\cos\gamma = \hat{r} \cdot \hat{n} = -(\cos\lambda \sin\psi \cos\delta + \sin\lambda \sin\delta)\cos\theta + \cos\lambda \cos\psi \sin\theta$$

This angle is equal to $\frac{\pi}{2}$ during dawn and dusk, while its cos reaches its highest at noon. In order to calculate φ we consider that $\cos(\varphi + \psi) = \hat{r}' \cdot \hat{n}'$. It is easy to find $\hat{n}'$:

Eq. 8 $$\hat{n}' = \frac{-\cos\lambda \sin\psi \hat{\varepsilon}_1' + \cos\lambda \cos\psi \hat{\varepsilon}_2'}{\sqrt{\cos^2\lambda \sin^2\psi + \cos^2\lambda \cos^2\psi}} = -\sin\psi\, \hat{\varepsilon}_1' + \cos\psi\, \hat{\varepsilon}_2'$$

As for $\hat{r}'$, we have to express $\hat{r}$ with respect to $\Omega; \xi', \eta', \zeta'$, so that we can consider its projection on the plane $\xi'\eta'$. Let's consider the expression with respect to $\Omega; \xi', \eta', \zeta$. From Eq. 4 we get:

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \\ \hat{\varepsilon}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \\ \hat{\varepsilon}_3 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \hat{r} = \cos\theta\, \hat{\varepsilon}_1 + \sin\theta\, \hat{\varepsilon}_2$$

On the other hand, from Eq. 2 we have

$$\begin{bmatrix} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \\ \hat{\varepsilon}_3 \end{bmatrix} = \begin{bmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & \cos\delta \end{bmatrix}^{-1} \begin{bmatrix} \hat{\varepsilon}'_1 \\ \hat{\varepsilon}'_2 \\ \hat{\varepsilon}'_3 \end{bmatrix} = \begin{bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} \begin{bmatrix} \hat{\varepsilon}'_1 \\ \hat{\varepsilon}'_2 \\ \hat{\varepsilon}'_3 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \hat{r} = \cos\theta\,(\cos\delta\, \hat{\varepsilon}'_1 - \sin\delta\, \hat{\varepsilon}'_3) + \sin\theta\, \hat{\varepsilon}'_2 = \cos\theta\cos\delta\, \hat{\varepsilon}'_1 + \sin\theta\, \hat{\varepsilon}'_2 - \cos\theta\sin\delta\, \hat{\varepsilon}'_3 \Rightarrow$$

Eq. 9 $$\hat{r}' = \frac{\cos\theta\cos\delta\hat{\varepsilon}'_1 + \sin\theta\hat{\varepsilon}'_2}{\sqrt{\cos^2\theta\cos^2\delta + \sin^2\theta}}$$

So, we can conclude that

Eq. 10 $$\cos(\varphi + \psi) = \hat{r}' \cdot \hat{n}' = \frac{\cos\psi\sin\theta - \sin\psi\cos\theta\cos\delta}{\sqrt{\cos^2\theta\cos^2\delta + \sin^2\theta}}$$

and also

Eq. 11 $$\cos(\varphi) = \hat{r}' \cdot \hat{\varepsilon}'_2 = \frac{\sin\theta}{\sqrt{\cos^2\theta\cos^2\delta + \sin^2\theta}}$$

1.5 Change in coordinates

In order to get a representation of the system in Figure 1 made by the computer, we have to express all the elements with respect to $O; x, y, z$. From Eq. 2 and Eq. 4 we get:

$$\begin{bmatrix} \hat{\varepsilon}'_1 \\ \hat{\varepsilon}'_2 \\ \hat{\varepsilon}'_3 \end{bmatrix} = \begin{bmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & \cos\delta \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} \Rightarrow$$

Eq. 12 $$\begin{bmatrix} \hat{\varepsilon}'_1 \\ \hat{\varepsilon}'_2 \\ \hat{\varepsilon}'_3 \end{bmatrix} = \begin{bmatrix} -\cos\delta & 0 & \sin\delta \\ 0 & -1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

Hence, if we indicate R the radius of the orbit around the Sun (we'll assume a circular orbit) and if ρ is the distance of a generic point Q from Ω, then the coordinates of Q with respect to $O; x, y, z$ can be calculated as follows:

$$\overrightarrow{\mathrm{OQ}} = \overrightarrow{\mathrm{O\Omega}} + \overrightarrow{\mathrm{\Omega Q}} = R\cos\theta\, \hat{e}_1 + R\sin\theta\, \hat{e}_2 + \rho\hat{n} =$$

$$= R\cos\theta\, \hat{e}_1 + R\sin\theta\, \hat{e}_2 - \rho\cos\lambda\sin\psi\, \hat{\varepsilon}'_1 + \rho\cos\lambda\cos\psi\, \hat{\varepsilon}'_2 + \rho\sin\lambda\, \hat{\varepsilon}'_3 =$$

$$= R\cos\theta\,\hat{e}_1 + R\sin\theta\,\hat{e}_2 - \rho\cos\lambda\sin\psi\,(-\cos\delta\,\hat{e}_1 + \sin\delta\,\hat{e}_3) + \rho\cos\lambda\cos\psi\,(-\hat{e}_2) +$$

$$+\rho\sin\lambda\,(\sin\delta\,\hat{e}_1 + \cos\delta\,\hat{e}_3) =$$

$$= R\cos\theta\,\hat{e}_1 + R\sin\theta\,\hat{e}_2 + \rho\cos\lambda\sin\psi\cos\delta\,\hat{e}_1 - \rho\cos\lambda\sin\psi\sin\delta\,\hat{e}_3 - \rho\cos\lambda\cos\psi\,\hat{e}_2 +$$

$$+\rho\sin\lambda\sin\delta\,\hat{e}_1 + \rho\sin\lambda\cos\delta\,\hat{e}_3 \Rightarrow$$

Eq. 13 $$\overrightarrow{\mathrm{OQ}} = \begin{bmatrix} \rho\cos\lambda\sin\psi\cos\delta + \rho\sin\lambda\sin\delta + R\cos\theta \\ R\sin\theta - \rho\cos\lambda\cos\psi \\ \rho\sin\lambda\cos\delta - \rho\cos\lambda\sin\psi\sin\delta \end{bmatrix}$$

By substituting in these coordinates ρ with the radius of the Earth (let's say ρ_E), we obtain the coordinates of P. For $\hat{n}'$ we have

$$\hat{n}' = -\frac{\cos\lambda}{|\cos\lambda|}\sin\psi\,\hat{\varepsilon}_1' + \frac{\cos\lambda}{|\cos\lambda|}\cos\psi\,\hat{\varepsilon}_2' =$$

$$= -\frac{\cos\lambda}{|\cos\lambda|}\sin\psi\,(-\cos\delta\,\hat{e}_1 + \sin\delta\,\hat{e}_3) - \frac{\cos\lambda}{|\cos\lambda|}\cos\psi\,\hat{e}_2 \Rightarrow$$

Eq. 14 $$\hat{n}' = \frac{\cos\lambda}{|\cos\lambda|}\sin\psi\cos\delta\,\hat{e}_1 - \frac{\cos\lambda}{|\cos\lambda|}\cos\psi\,\hat{e}_2 - \frac{\cos\lambda}{|\cos\lambda|}\sin\psi\sin\delta\,\hat{e}_3$$

1.6 Parallels and Equator

The parametric equations of the equator can be obtained from Eq. 13 by putting $\lambda = 0,\ \rho = \rho_E$, as ψ goes from zero to 2π:

Eq. 15 $$\mathcal{E}\colon \begin{cases} x = \rho_E\sin\psi\cos\delta + R\cos\theta \\ y = R\sin\theta - \rho_E\cos\psi \\ z = -\rho_E\sin\alpha\sin\delta \end{cases},\quad \psi \in [0,2\pi]$$

Similarly, the parallel corresponding to a latitude λ is:

Eq. 16 $$\mathcal{P}_\lambda\colon \begin{cases} x = \rho_E\cos\lambda\sin\psi\cos\delta + \rho_E\sin\lambda\sin\delta + R\cos\theta \\ y = R\sin\theta - \rho_E\cos\lambda\cos\psi \\ z = \rho_E\sin\lambda\cos\delta - \rho_E\cos\lambda\sin\psi\sin\delta \end{cases},\quad \psi \in [0,2\pi]$$

1.7 The half in darkness

What about the circumference that separates the half of the Earth which is in darkness from the other half? It can be found as the intersection between the plane defined by $\hat{r}$ (let's say π) and the sphere of the Earth. Let's consider the unit vector $\hat{v} = \sin\theta\,\hat{e}_1 - \cos\theta\,\hat{e}_2$ which is orthogonal with $\hat{r}$, then we can write $\pi\colon \overrightarrow{\mathrm{O\Omega}} + h\hat{e}_3 + k\hat{v}$, with $h, k \in \mathbb{R}$. So, we have:

Eq. 17 $$\pi\colon \begin{cases} x = R\cos\theta + k\sin\theta \\ y = R\sin\theta - k\cos\theta, \quad \forall\, h, k \in \mathbb{R} \\ z = h \end{cases}$$

To obtain the equation of the above-mentioned circumference (let's say $\mathcal{C}$), we must impose that $|h\hat{e}_3 + k\hat{v}| = \rho_E$:

$$\sqrt{k^2 + h^2} = \rho_E \Rightarrow k^2 + h^2 = \rho_E^2 \Rightarrow k = \pm\sqrt{\rho_E^2 - h^2} \Rightarrow$$

Eq. 18
$$\mathcal{C}_1: \begin{cases} x = R\cos\theta + \sqrt{\rho_E^2 - h^2}\sin\theta \\ y = R\sin\theta - \sqrt{\rho_E^2 - h^2}\cos\theta \\ z = h \end{cases}, \quad h \in [-\rho_E, \rho_E]$$

Eq. 19
$$\mathcal{C}_2: \begin{cases} x = R\cos\theta - \sqrt{\rho_E^2 - h^2}\sin\theta \\ y = R\sin\theta + \sqrt{\rho_E^2 - h^2}\cos\theta \\ z = h \end{cases}, \quad h \in [-\rho_E, \rho_E]$$

where $\mathcal{C}_1$ is half of the circumference $\mathcal{C}$ and $\mathcal{C}_2$ is the other half. More generally, we can find the equations of the half-sphere that is in darkness. We have $\pi: \overrightarrow{\mathrm{O}\Omega} - \hat{r}l + h\hat{e}_3 + k\hat{v}$, which means:

Eq. 20
$$\pi': \begin{cases} x = R\cos\theta + l\cos\theta + k\sin\theta \\ y = R\sin\theta + l\sin\theta - k\cos\theta, \quad \forall\, h, k \in \mathbb{R}, \forall\, l \in [0, \rho_E] \\ z = h \end{cases}$$

In this case, we have the condition:

$$\sqrt{k^2 + h^2} = (\rho_E - l) \Rightarrow k^2 + h^2 = (\rho_E - l)^2 \Rightarrow k = \pm\sqrt{(\rho_E - l)^2 - h^2} \Rightarrow$$

Eq. 21
$$\mathcal{C}_1': \begin{cases} x = (R + l)\cos\theta + \sqrt{(\rho_E - l)^2 - h^2}\sin\theta \\ y = (R + l)\sin\theta - \sqrt{(\rho_E - l)^2 - h^2}\cos\theta \\ z = h \end{cases}, \quad \begin{matrix} h \in [-(\rho_E - l), (\rho_E - l)] \\ l \in [0, \rho_E] \end{matrix}$$

Eq. 22
$$\mathcal{C}_2': \begin{cases} x = (R + l)\cos\theta - \sqrt{(\rho_E - l)^2 - h^2}\sin\theta \\ y = (R + l)\sin\theta + \sqrt{(\rho_E - l)^2 - h^2}\cos\theta \\ z = h \end{cases}, \quad \begin{matrix} h \in [-(\rho_E - l), (\rho_E - l)] \\ l \in [0, \rho_E] \end{matrix}$$

1.8 Dawn and dusk

In order to find the values of ψ (let's say $\psi_1$and ψ_2) for the dawn and the dusk at the latitude of choice, we have to find the two points that $\mathcal{C}_1$ and $\mathcal{C}_2$ have in common with $\mathcal{P}_\lambda$:

$$\mathcal{C}_1 \cap \mathcal{P}_\lambda: \begin{cases} \rho_E\cos\lambda\sin\psi\cos\delta + \rho_E\sin\lambda\sin\delta = \sqrt{\rho_E^2 - h^2}\sin\theta \\ \rho_E\cos\lambda\cos\psi = \sqrt{\rho_E^2 - h^2}\cos\theta \\ \rho_E(\sin\lambda\cos\delta - \cos\lambda\sin\psi\sin\delta) = h \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \cos\lambda\sin\psi\cos\delta + \sin\lambda\sin\delta = \sqrt{1-(\sin\lambda\cos\delta - \cos\lambda\sin\psi\sin\delta)^2}\sin\theta \\ \cos\lambda\cos\psi = \sqrt{1-(\sin\lambda\cos\delta - \cos\lambda\sin\psi\sin\delta)^2}\cos\theta \\ \rho_E(\sin\lambda\cos\delta - \cos\lambda\sin\psi\sin\delta) = h \end{cases}$$

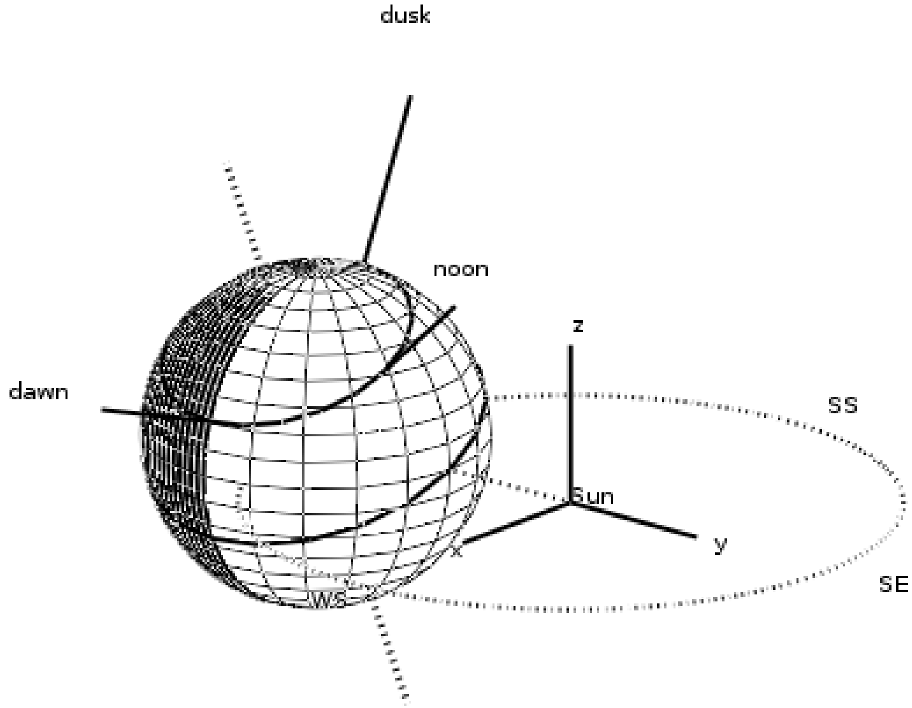


Figure 2. All the geometric elements discussed above are here plotted for the autumn equinox and for the latitude of Rome (about 42°). WS: winter solstice. SE: summer equinox. SS: summer solstice.

For $\theta \neq \pm\frac{\pi}{2}, \pi$ we have

$$\cos\lambda\sin\psi\cos\delta + \sin\lambda\sin\delta = \cos\psi\sin\theta\frac{\cos\lambda}{\cos\theta} \Rightarrow$$

Eq. 23 $$\cos\lambda(\cos\psi_1\sin\theta - \sin\psi_1\cos\theta\cos\delta) = \sin\lambda\cos\theta\sin\delta$$

In the same way, we find that

Eq. 24 $$\cos\lambda(\cos\psi_2\sin\theta - \sin\psi_2\cos\theta\cos\delta) = \sin\lambda\cos\theta\sin\delta$$

Now, considering Eq. 10, we have

Eq. 25 $$\cos(\varphi_1+\psi_1) = \cos(\varphi+\psi_2) = \frac{\sin\lambda\cos\theta\sin\delta}{\cos\lambda\sqrt{\cos^2\theta\cos^2\delta+\sin^2\theta}}, for\ \theta \neq \pm\frac{\pi}{2}, \lambda \neq \pm\frac{\pi}{2}$$

This relationship proofs that dusk and dawn are symmetric with respect to noon. It is easy to realize that Eq. 25 continues to be true for $\theta = \pm\frac{\pi}{2}$.

1.9 The codes

All that said, we can write the following two codes in MATLAB. The first one (CODE 1) plots frames like the one in Figure 2, one for each day of the year, and they can be uploaded in a movie editor (like for instance VSDC, a free source complete software for video editing) in order to build an animation of the rotation of planet Earth around the Sun.

CODE 1--

```
% file name = animation_3
% date of creation = 18/10/2019
% circular orbit
clear all
% conversion of units of measurments
N = 20;                                 % dots for the equator
R = 3.8;                                % radius of the orbit
ro_E = 1.3;                             % radius of the earth
lambda = 41.9109*pi/180;                % latitude (radiants)
delta = 23.45*pi/180;                   % tilt angle
theta(1) = 0;                           % winter solstice (21/22 December)
i_ws = 1;
day = 2*pi/365;
for i = 2:366
  theta(i) = theta(i-1) + day;
  if ( abs( theta(i) - (pi/2) ) <= day )
    i_se = i;                           % spring equinox (20 March)
  endif
  if ( abs( theta(i) - pi ) <= day )
    i_ss = i;                           % summer solstice (20/21 June)
  endif
  if ( abs( theta(i) - (3*pi/2) ) <= day )
    i_ae = i;                           % autumn equinox (22/23 September)
  endif
endfor
% angle psi
psi(1) = 0;
minute = pi/(12*60);
for i = 2:(24*60)+1
  psi(i) = psi(i-1) + minute;
endfor
for i = 1:366
  % axes
  figure (i)
  plot3 ([0, R/2], [0, 0], [0, 0], '-k', "linewidth", 2)
  text (R/2+R/15, 0, 0, "x")
  hold on
  plot3 ([0, 0], [0, R/2], [0, 0], '-k', "linewidth", 2)
  text (0, R/2+R/15, 0, "y")
  hold on
  plot3 ([0, 0], [0, 0], [0, R/2], '-k', "linewidth", 2)
  text (0, 0, R/2+R/15, "z")
  text (-0.2, -0.2, 0, "Sun")
  hold on
  % orbit
  X(1:366) = R*cos(theta(:));
  Y(1:366) = R*sin(theta(:));
  Z(1:366) = 0.;
  plot3 (X, Y, Z, ':k', "linewidth", 2)
  text ((R+R/5)*cos(theta(i_se)), (R+R/5)*sin(theta(i_se)), 0, "SE")
  text ((R+R/5)*cos(theta(i_ss)), (R+R/5)*sin(theta(i_ss)), 0, "SS")
  text ((R+R/5)*cos(theta(i_ae)), (R+R/5)*sin(theta(i_ae)), 0, "AE")
```

```
text ((R+R/5)*cos(theta(i_ws)), (R+R/5)*sin(theta(i_ws)), 0, "WS")
hold on
colormap ([0,0,0])
[x, y, z] = ellipsoid (R*cos(theta(i)), R*sin(theta(i)), 0., ro_E, ro_E, ro_E, N);
mesh(x, y, z);
axis equal;
hold on
% vector r
plot3 ([0, R*cos(theta(i))], [0, R*sin(theta(i))], [0, 0], ':k', "linewidth", 2)
hold on
% the Erath's axis of revolution
plot3 ([R*cos(theta(i)), R*cos(theta(i))+2*ro_E*sin(delta)], [R*sin(theta(i)),R*sin(theta(i))], \
[0, 2*ro_E*cos(delta)], ':k', "linewidth", 3)
hold on
plot3 ([R*cos(theta(i)), R*cos(theta(i))-2*ro_E*sin(delta)], [R*sin(theta(i)), R*sin(theta(i))], \
[0, -2*ro_E*cos(delta)], ':k', "linewidth", 3)
% the equator
alpha = [0:(2*pi/N):2*pi];
X = ro_E*sin(alpha)*cos(delta) + R*cos(theta(i));
Y = R*sin(theta(i)) - ro_E*cos(alpha);
Z = -ro_E*sin(alpha)*sin(delta);
plot3 (X, Y, Z, '-k', "linewidth", 2)
hold on
% parallel of choice
ro = ro_E;
X = ro*cos(lambda)*sin(alpha)*cos(delta) + ro*sin(lambda)*sin(delta) + R*cos(theta(i));
Y = R*sin(theta(i)) - ro*cos(lambda)*cos(alpha);
Z = ro*sin(lambda)*cos(delta) - ro*cos(lambda)*sin(alpha)*sin(delta);
plot3 (X, Y, Z, '-k', "linewidth", 2)
hold on
% shadow of the night
l = [0:(ro_E/N):ro_E];
for t=1:N
  ro = sqrt( (ro_E^2) - (l(t)^2) );
  h = [-ro:(2*ro/N):ro];
  for s=1:N+1
    X(s) = (R+l(t))*cos(theta(i)) + sin( theta(i) )*sqrt( (ro^2) - (h(s)^2) );
    Y(s) = (R+l(t))*sin(theta(i)) - cos( theta(i) )*sqrt( (ro^2) - (h(s)^2) );
    Z(s) = h(s);
  endfor
  plot3 (X, Y, Z, '-k', "linewidth", 2)
  hold on
  for s=1:N+1
    X(s) = (R+l(t))*cos(theta(i)) - sin( theta(i) )*sqrt( (ro^2) - (h(s)^2) );
    Y(s) = (R+l(t))*sin(theta(i)) + cos( theta(i) )*sqrt( (ro^2) - (h(s)^2) );
    Z(s) = h(s);
  endfor
  plot3 (X, Y, Z, '-k', "linewidth", 2)
endfor
% scalar product between n and r
for j=1:(24*60) + 1
  % unit vector n
  ro = ro_E;
  X_n (1) = ro*cos(lambda)*sin(psi(j))*cos(delta) + ro*sin(lambda)*sin(delta) + R*cos(theta(i));
  Y_n (1) = R*sin(theta(i)) - ro*cos(lambda)*cos(psi(j));
  Z_n (1) = ro*sin(lambda)*cos(delta) - ro*cos(lambda)*sin(psi(j))*sin(delta);
  ro = ro_E + ro_E/2;
```

```
  X_n (2) = ro*cos(lambda)*sin(psi(j))*cos(delta) + ro*sin(lambda)*sin(delta) + R*cos(theta(i));
  Y_n (2) = R*sin(theta(i)) - ro*cos(lambda)*cos(psi(j));
  Z_n (2) = ro*sin(lambda)*cos(delta) - ro*cos(lambda)*sin(psi(j))*sin(delta);
  scalar_p(j) = [cos(lambda)*sin(psi(j))*cos(delta) + sin(lambda)*sin(delta)]*( -cos(theta(i)) )+\
 [cos(lambda)*cos(psi(j))]*( sin(theta(i)) );
endfor
% noon, dawn and dusk
for j=1:(24*60) + 1
  if ( ( scalar_p(j) ) == ( max( scalar_p ) ) )
    j_noon = j;
    psi_noon (i) = psi(j);
  endif
  if ( abs( scalar_p(j) ) ==  min( abs(scalar_p) )  )
    j_1 = j;
  endif
endfor
for j=1:(24*60) + 1
  if ( abs( scalar_p(j) ) ==  min( abs( [scalar_p(1:j_1-1), scalar_p(j_1+1:(24*60) + 1)] ) ) )
    j_2 = j;
  endif
endfor
% angle (fi + psi) at dawn and at dusk
cos_fi_psi_num = sin(lambda)*cos(theta(i))*sin(delta);
cos_fi_psi_den = cos(lambda)*sqrt( [( cos(theta(i))*cos(delta) )^2] + [sin(theta(i))^2] )
cos_fi_psi = cos_fi_psi_num/cos_fi_psi_den;
fi_psi_dusk = acos( cos_fi_psi );
fi_psi_dawn = - acos( cos_fi_psi );
% the timing of dusk and dawn
h_dawn(i) = 12 + ( ( ( fi_psi_dawn )/minute )/60 );
h_dusk(i) = 12 + ( ( ( fi_psi_dusk )/minute )/60 );
% plotting of unit vector n at dawn, noon, and dusk
% unit vector n noon
ro = ro_E;
X_n (1) = ro*cos(lambda)*sin(psi(j_noon))*cos(delta) + ro*sin(lambda)*sin(delta) + R*cos(theta(i));
Y_n (1) = R*sin(theta(i)) - ro*cos(lambda)*cos(psi(j_noon));
Z_n (1) = ro*sin(lambda)*cos(delta) - ro*cos(lambda)*sin(psi(j_noon))*sin(delta);
ro = ro_E + ro_E;
X_n (2) = ro*cos(lambda)*sin(psi(j_noon))*cos(delta) + ro*sin(lambda)*sin(delta) + R*cos(theta(i));
Y_n (2) = R*sin(theta(i)) - ro*cos(lambda)*cos(psi(j_noon));
Z_n (2) = ro*sin(lambda)*cos(delta) - ro*cos(lambda)*sin(psi(j_noon))*sin(delta);
plot3 (X_n, Y_n, Z_n, '-k', "linewidth", 2)
text (1.2*X_n(2), 1.2*Y_n(2), 1.2*Z_n(2), "noon ")
% unit vector n_1
ro = ro_E;
X_n (1) = ro*cos(lambda)*sin(psi(j_1))*cos(delta) + ro*sin(lambda)*sin(delta) + R*cos(theta(i));
Y_n (1) = R*sin(theta(i)) - ro*cos(lambda)*cos(psi(j_1));
Z_n (1) = ro*sin(lambda)*cos(delta) - ro*cos(lambda)*sin(psi(j_1))*sin(delta);
ro = ro_E + ro_E;
X_n (2) = ro*cos(lambda)*sin(psi(j_1))*cos(delta) + ro*sin(lambda)*sin(delta) + R*cos(theta(i));
Y_n (2) = R*sin(theta(i)) - ro*cos(lambda)*cos(psi(j_1));
Z_n (2) = ro*sin(lambda)*cos(delta) - ro*cos(lambda)*sin(psi(j_1))*sin(delta);
plot3 (X_n, Y_n, Z_n, '-k', "linewidth", 2)
% unit vector n_2
ro = ro_E;
X_n (1) = ro*cos(lambda)*sin(psi(j_2))*cos(delta) + ro*sin(lambda)*sin(delta) + R*cos(theta(i));
Y_n (1) = R*sin(theta(i)) - ro*cos(lambda)*cos(psi(j_2));
Z_n (1) = ro*sin(lambda)*cos(delta) - ro*cos(lambda)*sin(psi(j_2))*sin(delta);
```

```
  ro = ro_E + ro_E;
  X_n (2) = ro*cos(lambda)*sin(psi(j_2))*cos(delta) + ro*sin(lambda)*sin(delta) + R*cos(theta(i));
  Y_n (2) = R*sin(theta(i)) - ro*cos(lambda)*cos(psi(j_2));
  Z_n (2) = ro*sin(lambda)*cos(delta) - ro*cos(lambda)*sin(psi(j_2))*sin(delta);
  plot3 (X_n, Y_n, Z_n, '-k', "linewidth", 2)
  save the frame
  name = ["frame", int2str(i),".jpg"];
  print (figure(i), name)
endfor
h_dawn(i)
h_dusk (i)
```

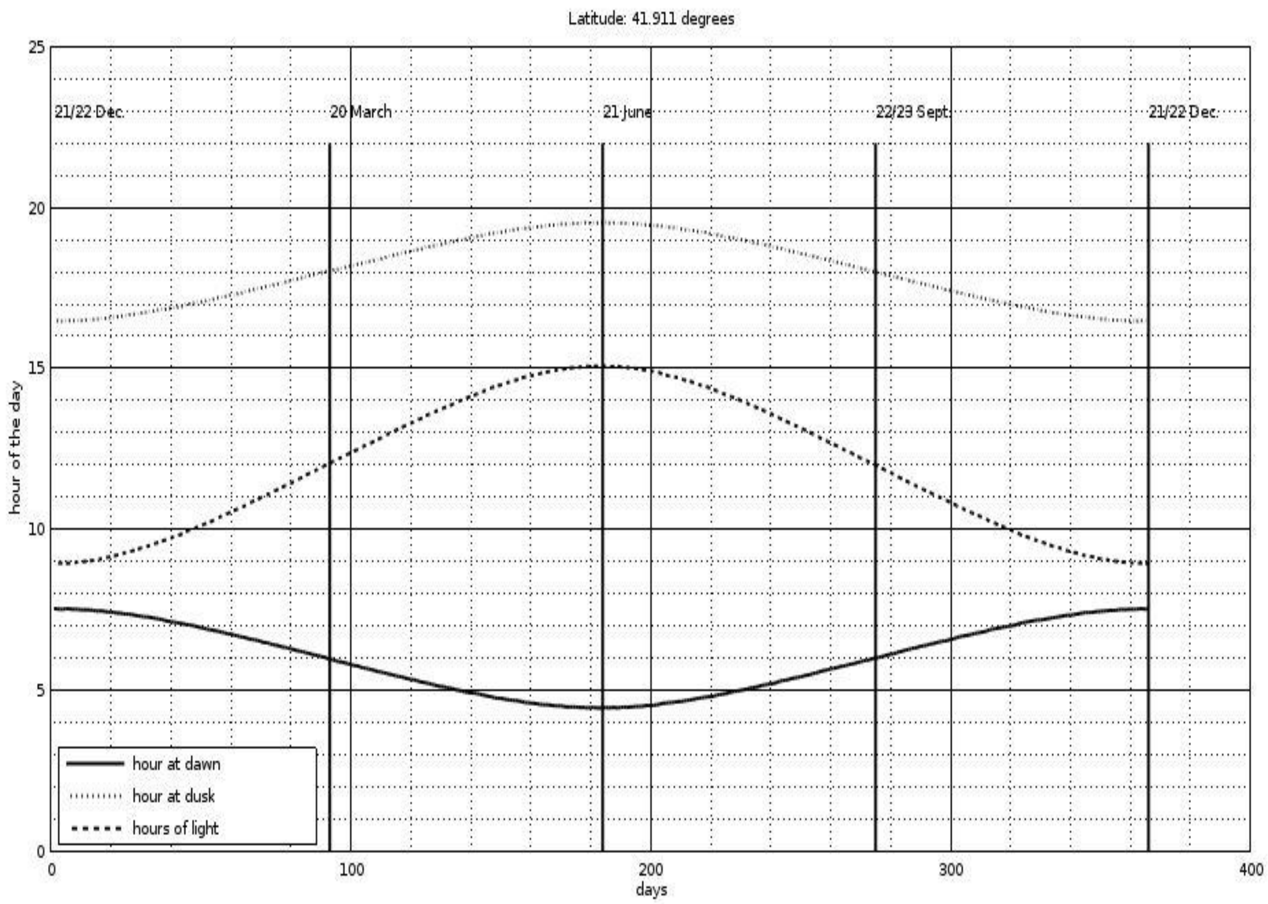


Figure 3. Hours of dawn and dusk for each day of the year, for a latitude of about 42°. The length of the day is also plotted.

The second plots the hour at dawn, the hour at dusk, the hours of light per day, and the height of the sun on the horizon, for each day of the year at the latitude specified by the user. In Figure 3 you find the hour of dawn and dusk and the length of the day for each day of the year, for a latitude of about 42°. In Figure 4, the height of the sun on the horizon is plotted, for the same latitude. In Figure 5 you find the hour of dawn and dusk and the length of the day for each day of the year, for a latitude of about -32°. In Figure 6, the height of the sun on the horizon is plotted, for the same latitude.

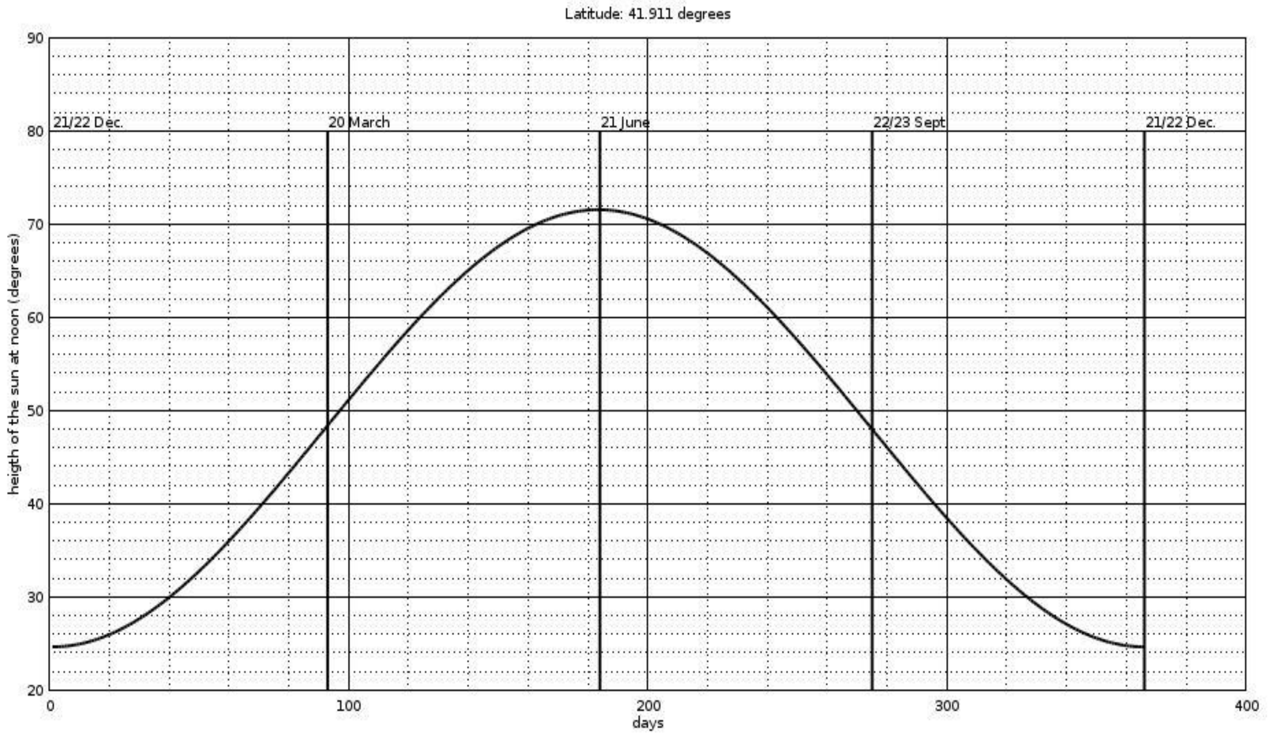


Figure 4. Height of the sun on the horizon for each day of the year at a latitude of about 42°.

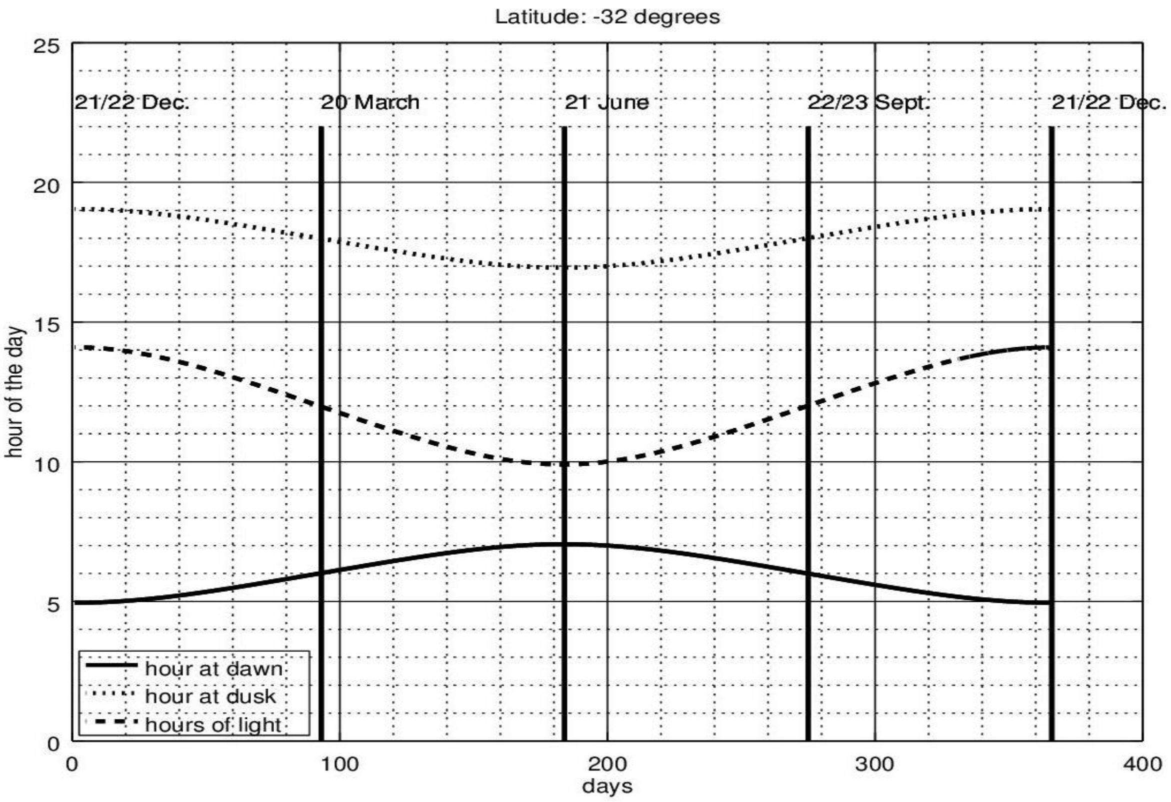


Figure 5. Hours of dawn and dusk for each day of the year, for a latitude of about -32°. The length of the day is also plotted.

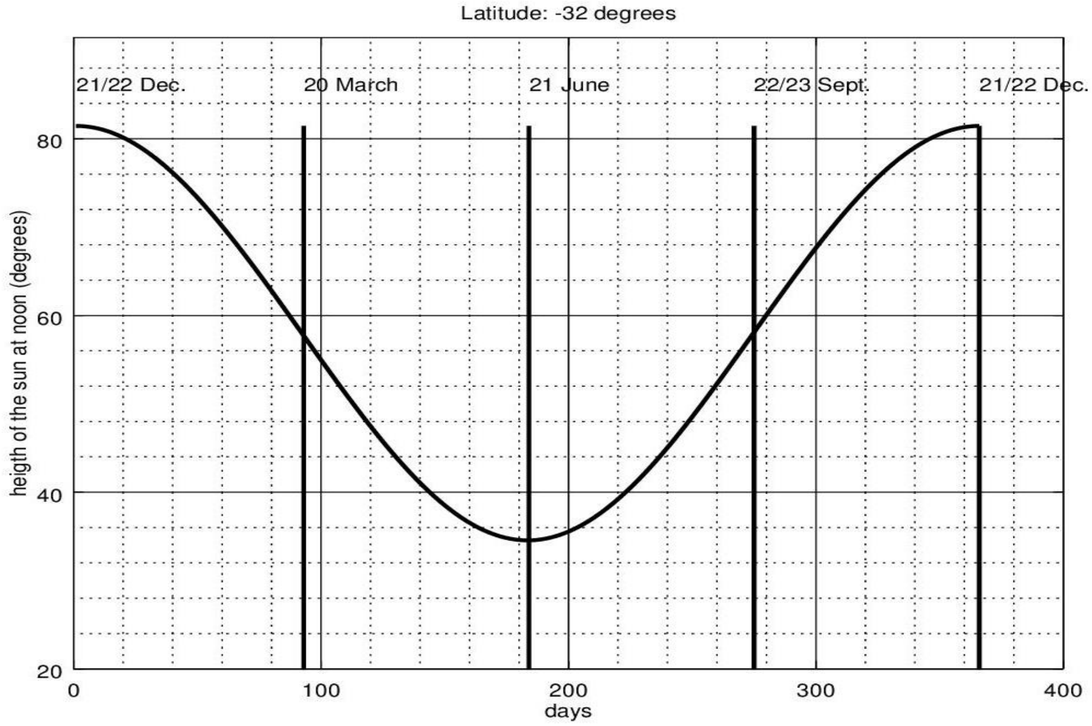


Figure 6. Height of the sun on the horizon for each day of the year at a latitude of about -32°.

CODE 2---

```
% file name = till_dusk_2
% date of creation = 14/10/2019
% circular orbit
clear all
% conversion of units of measurments
N = 20;                                       % dots for the equator
R = 3.8;                                      % radius of the orbit
ro_E = 1.3;                                   % radius of the earth
lambda = 41.9109*pi/180;                      % latitude (radiants)
delta = 23.45*pi/180;                         % tilt angle
theta(1) = 0;                                 % winter solstice (21/22 December)
i_ws = 1;
day = 2*pi/365;
days = [1:1:366];
for i = 2:366
  theta(i) = theta(i-1) + day;
  if ( abs( theta(i) - (pi/2) ) <= day )
    i_se = i;                                 % spring equinox (20 March)
  endif
  if ( abs( theta(i) - pi ) <= day )
    i_ss = i;                                 % summer solstice (20/21 June)
  endif
  if ( abs( theta(i) - (3*pi/2) ) <= day )
    i_ae = i;                                 % autumn equinox (22/23 September)
  endif
endfor
% angle psi
psi(1) = 0;
minute = pi/(12*60);
```

```
for i = 2:(24*60)+1
  psi(i) = psi(i-1) + minute;
endfor
% hours
index = 366;
for i=1:index
  for j=1:(24*60) + 1
    % unit vector n
    ro = ro_E;
    X_n (1) = ro*cos(lambda)*sin(psi(j))*cos(delta) + ro*sin(lambda)*sin(delta) + R*cos(theta(i));
    Y_n (1) = R*sin(theta(i)) - ro*cos(lambda)*cos(psi(j));
    Z_n (1) = ro*sin(lambda)*cos(delta) - ro*cos(lambda)*sin(psi(j))*sin(delta);
    ro = ro_E + ro_E/2;
    X_n (2) = ro*cos(lambda)*sin(psi(j))*cos(delta) + ro*sin(lambda)*sin(delta) + R*cos(theta(i));
    Y_n (2) = R*sin(theta(i)) - ro*cos(lambda)*cos(psi(j));
    Z_n (2) = ro*sin(lambda)*cos(delta) - ro*cos(lambda)*sin(psi(j))*sin(delta);
    % scalar product between n and r
    scalar_p(j) = [cos(lambda)*sin(psi(j))*cos(delta) + sin(lambda)*sin(delta)]*( -cos(theta(i)) )\
    + [(-1)*cos(lambda)*cos(psi(j))]*( -sin(theta(i)) );
  endfor
  % value of psi at noon
  for j=1:(24*60) + 1
    if ( ( scalar_p(j) ) == ( max( scalar_p ) ) )
      j_noon = j;
      psi_noon (i) = psi(j);
    endif
  endfor
  % angle (fi + psi) at dawn and at dusk
  cos_fi_psi_num = sin(lambda)*cos(theta(i))*sin(delta);
  cos_fi_psi_den = cos(lambda)*sqrt( [( cos(theta(i))*cos(delta) )^2] + [sin(theta(i))^2] );
  cos_fi_psi = cos_fi_psi_num/cos_fi_psi_den;
  fi_psi_dusk = acos( cos_fi_psi );
  fi_psi_dawn = - acos( cos_fi_psi );
  % the timing of dusk and dawn
  h_dawn(i) = 12 + ( ( ( fi_psi_dawn )/minute )/60 );
  h_dusk(i) = 12 + ( ( ( fi_psi_dusk )/minute )/60 );
  % hours of light
  h_light (i) = h_dusk(i) - h_dawn(i);
  % angle between n and r at noon
  angle_noon (i) = acos( scalar_p(j_noon) )*180/pi;
  % heigth of the sun at noon
  angle_noon_2 (i) = 90 - angle_noon (i);
endfor
% title
tit = ["Latitude: ", num2str(lambda*180/pi)," degrees"];
h_dusk(21)= ( h_dusk (22) + h_dusk (20) )/2;
h_dawn(21) = ( h_dawn(22) + h_dawn(20) )/2;
h_light (21) = h_dusk(21) - h_dawn(21);
% plotting
figure (2)
plot (days(1:index), h_dawn(1:index), '-k', "linewidth", 2)
hold on
plot (days(1:index), h_dusk(1:index), ':k', "linewidth", 2)
hold on
plot (days(1:index), h_light(1:index), '--k', "linewidth", 2)
grid minor on
xlabel('days');
```

```
ylabel('hour of the day');
legend ('hour at dawn', 'hour at dusk', 'hours of light',"location",'SOUTHWEST')
hold on
plot ([i_se,i_se], [0,22], '-k', "linewidth", 2)
plot ([i_ss,i_ss], [0,22], '-k', "linewidth", 2)
plot ([i_ae,i_ae], [0,22], '-k', "linewidth", 2)
plot ([366,366], [0,22], '-k', "linewidth", 2)
title (tit)
text (1, 23, "21/22 Dec.")
text (i_se, 23, "20 March")
text (i_ss, 23, "21 June")
text (i_ae, 23, "22/23 Sept.")
text (366, 23, "21/22 Dec.")
figure (3)
plot (days(1:index), angle_noon_2 (1:index), '-k', "linewidth", 2)
grid minor on
xlabel('days');
ylabel('heigth of the sun at noon (degrees)');
axis ([0,400, 20, 90])
hold on
plot ([i_se,i_se], [0,80], '-k', "linewidth", 2)
plot ([i_ss,i_ss], [0,80], '-k', "linewidth", 2)
plot ([i_ae,i_ae], [0,80], '-k', "linewidth", 2)
plot ([366,366], [0,80], '-k', "linewidth", 2)
hold on
text (1, 81, "21/22 Dec.")
text (i_se, 81, "20 March")
text (i_ss, 81, "21 June")
text (i_ae, 81, "22/23 Sept.")
text (366, 81, "21/22 Dec.")
title (tit)
------------------------------------------------------------------------------------------------------------------
```

2 Solar radiation, a theoretical model

2.1 Introduction

In this chapter, we will describe the physical properties of the electromagnetic radiation that our planet receives from the Sun. We shall study this radiation just outside the atmosphere, as a first step. We shall then add the effect of the atmosphere, which depends on the angle γ introduced in Eq. 7. At the end of this work, we shall dispose of a code that gives us the power, per unit area and unit wavelength, that the Sun deliver at sea level at noon, in function of the latitude, the day of the year, and the wavelength. In the present chapter, we rely mainly on mathematical models for all the physical phenomena involved.
But, if we have empirical data, why wasting our time in building a mathematical model? It is not a waste: building a mathematical model will be a way to better understand the mechanism behind this phenomenon. Even if we are likely going to discard it, in favour of the empirical one, we are going to learn a lot about solar radiation on planet Earth in the process.

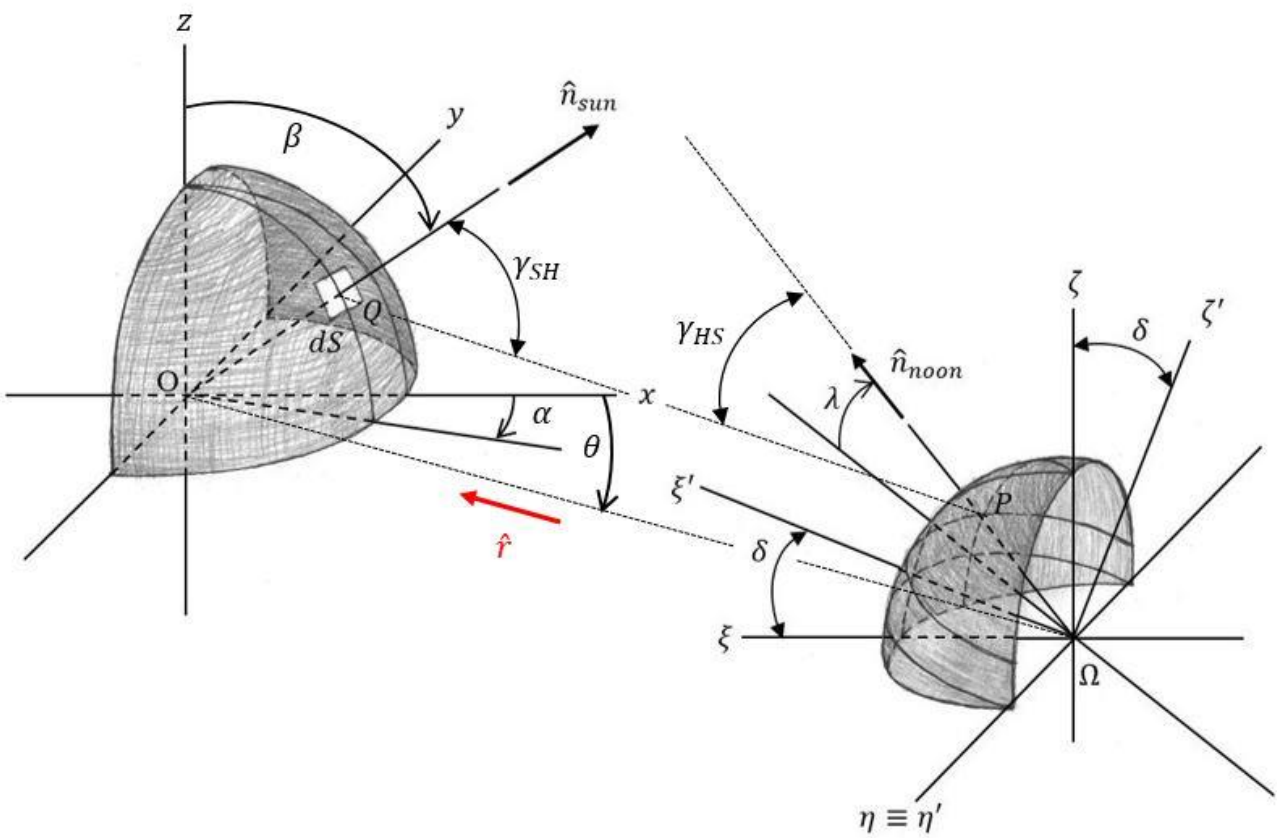


Figure 7. Symbology used in this paper for the transmission of radiant energy from Sun to Earth.

2.2 Angular monochromatic emissive power

Bodies can exchange energy from one to the other by means of electromagnetic waves. This form of energy emitted by bodies is called radiant energy. For a complete discussion of radiant energy you can read (1) or (2); we are here interested only in those definitions and concepts strictly necessary to elaborate a model for the radiant energy emitted by the Sun.
Let's consider the surface dS on the Sun (Figure 7), an elementary surface around point Q whose coordinates with respect to $O; x, y, z$ are $(R_{Sun} \cos\beta \cos\alpha, R_{Sun} \cos\beta \sin\alpha, R_{Sun} \sin\beta)$. Let's

indicate $dW\left(L, \alpha, \beta, \hat{r}_{QP}\right)$The power emitted by it along the direction QP ($\hat{r}_{QP}$ is the unit vector of $\overrightarrow{QP}$). Then, we define *angular monochromatic emissive power* of dS the following quantity:

Eq. 26 $$\sigma_S\left(L, \alpha, \beta, \hat{r}_{QP}, T\right) = \frac{dW\left(L,\alpha,\beta,\hat{r}_{QP},T\right)}{d\Omega \cdot dL \cdot dS \cdot cos(\gamma_{SH})}$$

where $dS \cdot cos(\gamma_{SH})$ is the surface orthogonal with respect to $\overrightarrow{QP}$, T is the absolute temperature of the surface of the Sun, while $d\Omega$ is the solid angle within which we are considering the emission. We can assume, as a simplification, that whatever the values α, β are, the emission is the same. So, we have:

Eq. 27 $$\sigma_S\left(L, \hat{r}_{QP}, T\right) = \frac{dW\left(L,\hat{r}_{QP},T\right)}{d\Omega \cdot dL \cdot dS \cdot cos(\gamma_{SH})}$$

If for instance, we consider the emission by dS that hits a surface dS_P around point P, we have for the solid angle:

Eq. 28 $$d\Omega = \frac{dS_P \cdot cos(\gamma_{HS})}{\overline{QP}^2}$$

2.3 Monochromatic emissive power

The emission within the hemispace in front of dS is called *monochromatic emissive power* and it is indicated $\varepsilon_S(L, \alpha, \beta)$. It is given by:

Eq. 29 $$\varepsilon_S(L, T) = \int_{semispace} \sigma_S\left(L, \hat{r}_{QP}, T\right) cos(\gamma_{SH}) d\Omega$$

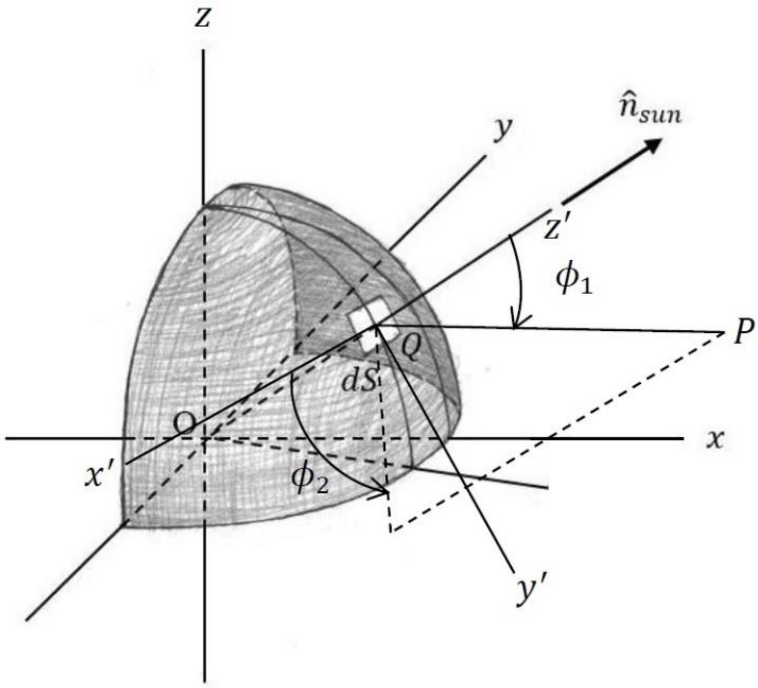


Figure 8. Mobile coordinate system, centred in Q.

In order to calculate this integral, we need a further coordinate system, centred in Q (Figure 8). We have:

$$d\Omega = sin\phi_1 d\phi_1 d\phi_2 \Rightarrow \varepsilon_S(L,T) = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sigma_S(L,\hat{r}_{QP},T) cos\phi_1 sin\phi_1 d\phi_1 d\phi_2$$

Assuming that the emission from dS is the same in each direction, we can write:

$$\varepsilon_S(L,T) = \sigma_S(L,T)\int_0^{2\pi}\int_0^{\frac{\pi}{2}} cos\phi_1 sin\phi_1 d\phi_1 d\phi_2 = -2\pi\sigma_S(\lambda,T)\int_0^{\frac{\pi}{2}} cos\phi_1 d(cos\phi_1) \Rightarrow$$

Eq. 30 $\qquad \varepsilon_S(L,T) = \pi\sigma_S(L,T)$

2.4 Sun emissive power outside the atmosphere

Now, assuming that the Sun is a *black* body, according to *Plank's law* we have

Eq. 31 $\qquad \varepsilon_S(L) = \frac{C_1}{L^5\left(e^{\frac{C_2}{LT_S}}-1\right)}$

where T is the absolute temperature of the surface of the Sun, and where $C_1 = 3.7415 \cdot 10^{-16} Wm^2$ and $C_2 = 1.4388 \cdot 10^{-2} mK$ (2). For T we can assume a value of $T_S = 5875\,K$ (3). If we now consider Eq. 27, Eq. 28, Eq. 31, we have

$$\frac{C_1}{\pi L^5\left(e^{\frac{C_2}{LT_S}}-1\right)} = \frac{dW(L)}{\frac{dS_P \cdot cos(\gamma_{HS})}{\overline{QP}^2}\cdot dL \cdot dS \cdot cos(\gamma_{SH})} \Leftrightarrow$$

Eq. 32 $\qquad dW(L) = \frac{C_1 cos(\gamma_{SH})}{\pi L^5\left(e^{\frac{C_2}{LT_S}}-1\right)\overline{QP}^2} dS_P cos(\gamma_{HS}) dL dS$

Now, given the huge distance between the Sun and planet Earth, with respect to the diameter of both the Sun and the earth, we can assume that $cos(\gamma_{HS}) = \hat{r}\cdot\hat{n}_{noon}$ and $cos(\gamma_{SH}) = -\hat{r}\cdot\hat{n}_{sun}$. Moreover, we can assume that $\overline{QP}$ is the distance between Sun and Earth ($\overline{O\Omega}$). So we have:

Eq. 33 $\qquad \frac{dW(L)}{(\hat{r}\cdot\hat{n}_{noon})dS_P dL} = \frac{C_1(-\hat{r}\cdot\hat{n}_{sun})}{\pi L^5\left(e^{\frac{C_2}{LT_S}}-1\right)\overline{O\Omega}^2} dS$

We have now to integrate on the surface of the Sun that faces the Earth (which is half a sphere). The simplest way to do that is to calculate $\theta = 0$, given that the result is the same for each day of the year. We have:

$$\frac{W(L)}{dS_P(\hat{r}\cdot\hat{n}_{noon})dL} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi} \frac{C_1(-\hat{r}\cdot\hat{n}_{sun})}{\pi L^5\left(e^{\frac{C_2}{LT_S}}-1\right)\overline{QP}^2} R_S^2 \sin\beta\, d\beta d\alpha =$$

$$= -\frac{C_1 R_S^2}{\pi L^5\left(e^{\frac{C_2}{LT_S}}-1\right)\overline{O\Omega}^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi} \hat{r}\cdot\hat{n}_{sun} \sin\beta\, d\beta d\alpha$$

But we have:

$$\hat{r}\cdot\hat{n}_{sun} = (-\hat{e}_1)\cdot(\sin\beta\cos\alpha\,\hat{e}_1 + \sin\beta\sin\alpha\,\hat{e}_2 + cos\beta\hat{e}_3) = -\sin\beta\cos\alpha$$

Hence, we can write:

$$\frac{W(L)}{dS_P(\hat{r}\cdot\hat{n}_{noon})dL} = \frac{C_1R_S^2}{\pi L^5\left(e^{\frac{C_2}{LT_S}}-1\right)\overline{O\Omega}^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\alpha\,d\alpha\int_0^{\pi}\sin^2\beta\,d\beta$$

Let's consider now that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\alpha\,d\alpha = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\sin\alpha = 2$$

$$\int_0^{\pi}\sin^2\beta\,d\beta = \int_0^{\pi}\frac{1-cos(2\beta)}{2}d\beta = \frac{1}{2}\left(\pi-\frac{1}{2}\int_0^{\pi}d(sin(2\beta))\right) = \frac{1}{2}\left(\pi-\frac{1}{2}[sin(2\beta)]_0^{\pi}\right) = \frac{\pi}{2}$$

So, we have

Eq. 34 $$I(L) \triangleq \frac{W(L)}{(\hat{r}\cdot\hat{n}_{noon})dS_PdL} = \frac{C_1R_S^2}{L^5\left(e^{\frac{C_2}{LT_S}}-1\right)\overline{O\Omega}^2}$$

where $R_S = 696\cdot10^6m$ is the radius of the Sun at the level of its photosphere. As for $\overline{O\Omega}$, it is a function of θ, since the orbit of the Earth around the Sun is an ellipse (Figure 9) whose parametric equation, with respect to $O';x',y',z'$ is:

Eq. 35 $$\overline{O\Omega}(\theta') = \frac{1}{\frac{G\mathcal{M}_S\mathcal{M}_H^2}{l^2}+sin(\theta)\sqrt{\frac{G^2\mathcal{M}_S^2\mathcal{M}_H^2}{l^4}+\frac{2\mathcal{M}_HE}{l^2}}}$$

where $\mathcal{M}_S$is the mass of the Sun, $\mathcal{M}_H$is the mass of the Earth, G is the gravitational parameter, l is the angular momentum of $\mathcal{M}_H$ and E is its total energy. The orbit can be derived from the gravitational law, of course (4). Now, instead of calculating l and E, we can just consider these two conditions:

$$\begin{cases}\overline{O\Omega}\left(\frac{\pi}{2}\right) = 147.10\cdot10^6\,km\\ \overline{O\Omega}\left(-\frac{\pi}{2}\right) = 152.10\cdot10^6\,km\end{cases} \Leftrightarrow \begin{cases}\frac{1}{A+B} = 147.10\cdot10^6\,km\\ \frac{1}{A-B} = 152.10\cdot10^6\,km\end{cases} \Leftrightarrow \begin{cases}B = 1.12\cdot10^{-10}km^{-1}\\ A = 6.69\cdot10^{-9}km^{-1}\end{cases} \Rightarrow$$

Eq. 36 $$\overline{O\Omega}(\theta') = \frac{1}{\frac{6.69}{10^9km}+\frac{1.12}{10^{10}km}sin(\theta')}$$

But we have $\theta' = \theta+\Delta$ and also $\theta = 13\frac{2\pi}{365}$ when $\theta' = \frac{\pi}{2}$ and also that $\theta' = \theta+\Delta$, so we can conclude that

$$\frac{\pi}{2} = 13\frac{2\pi}{365}+\Delta \Rightarrow \Delta = \frac{\pi}{2}-13\frac{2\pi}{365} = \pi\left(\frac{1}{2}-\frac{26}{365}\right) = \pi\left(\frac{365-52}{730}\right) = \pi\frac{313}{730} \Rightarrow$$

Eq. 37 $$\overline{O\Omega}(\theta) = \frac{1}{\frac{6.69}{10^9 km} + \frac{1.12}{10^{10} km} sin\left(\theta + \pi\frac{313}{730}\right)}$$

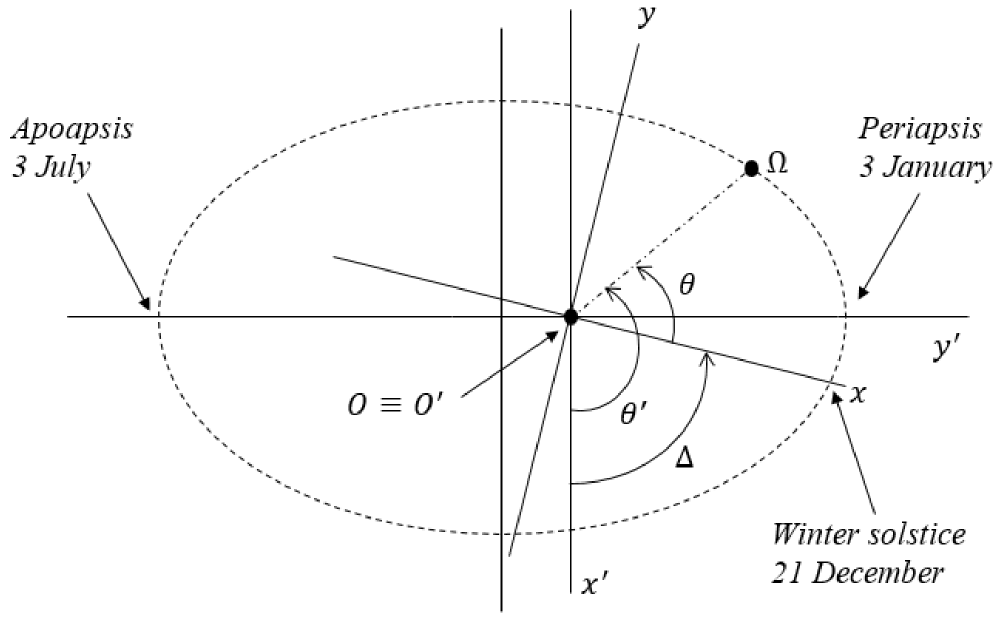


Figure 9. The orbit of the Earth around the Sun is an ellipse with the Sun in one of the two focal points. The eccentricity of the orbit has been exaggerated in this picture.

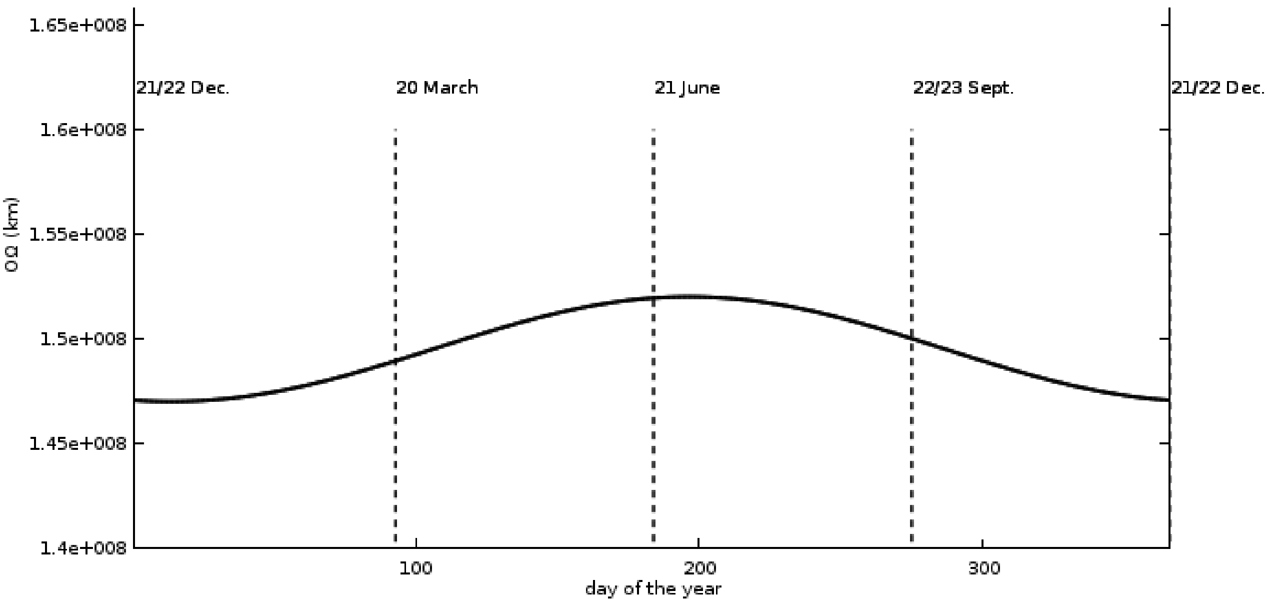


Figure 10. The distance between the Earth and the Sun in function of the day of the year.

The diagram of this function is reported in Figure 10. By substituting $\overline{O\Omega}(\theta)$ in Eq. 34 we obtain the expression of the emission, per unit of orthogonal area, per unit wavelength, outside the atmosphere, in function of the day of the year and the wavelength:

Eq. 38 $$I(L,\theta) \triangleq \frac{W(L)}{(\hat{r}\cdot\hat{n}_{noon})dS_P dL} = \frac{C_1 R_S^2}{L^5\left(e^{\frac{C_2}{\lambda T_S}}-1\right)\left(\frac{1}{\frac{6.69}{10^9 km}+\frac{1.12}{10^{10} km}sin\left(\theta+\pi\frac{313}{730}\right)}\right)^2}$$

The diagram for this function is reported in Figure 11 for both the winter solstice and the summer solstice.

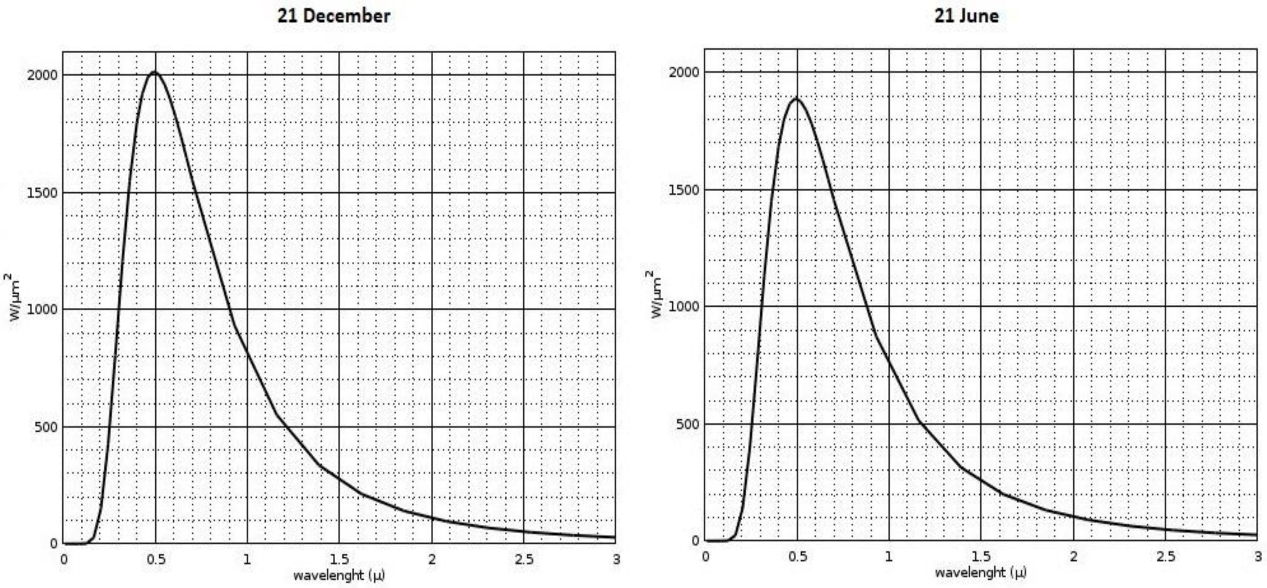


Figure 11. Radiant emission of the Sun, per unit of orthogonal area, per unit wavelength, outside the atmosphere in correspondence of winter solstice (left) and of the summer solstice (right).

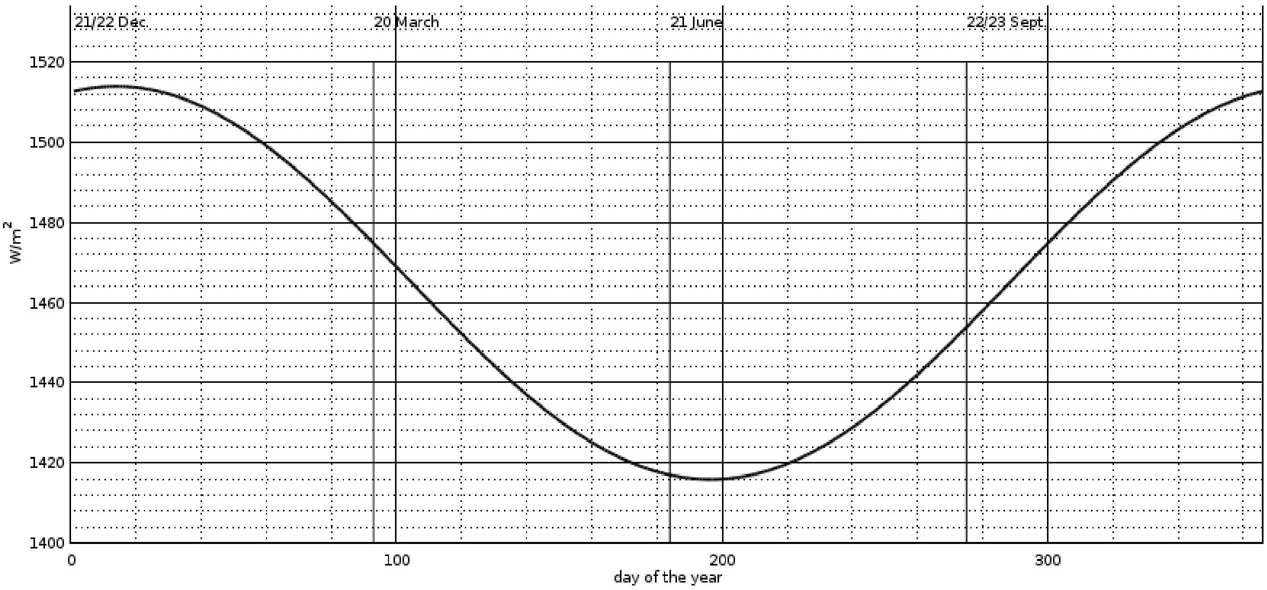


Figure 12. Radiant emission of the Sun, per unit of an orthogonal area outside the atmosphere in function of the day of the year.

As for the total power emitted per unit area by the Sun outside the atmosphere, we have to integrate Eq. 34 with respect to L. According to the *Stefan-Boltzmann law* (see 3.1) we have

Eq. 39 $$\int_0^{\infty} \frac{C_1}{L^5\left(e^{\frac{C_2}{LT_S}}-1\right)} dL = 5.670 \cdot 10^{-8} \frac{W}{m^2K^4} T_S^4$$

Thus, we can write

$$J_S = \int_0^\infty \frac{C_1 R_S^2}{L^5\left(e^{\frac{C_2}{LT_S}} - 1\right)\overline{O\Omega}^2} d\lambda = \frac{R_S^2}{\overline{O\Omega}^2}\int_0^\infty \frac{C_1}{L^5\left(e^{\frac{C_2}{LT_S}} - 1\right)} dL = \frac{R_S^2 T_S^4}{\overline{O\Omega}^2} 5.670 \cdot 10^{-8} \frac{W}{m^2K^4} \Rightarrow$$

Eq. 40 $$J_S(\theta) = \frac{R_S^2 T_S^4}{\overline{O\Omega}^2(\theta)} 5.670 \cdot 10^{-8} \frac{W}{m^2K^4} = \frac{1.614 \cdot 10^{14} W}{\overline{O\Omega}^2(\theta)}$$

The diagram of $J_S(\theta)$ is the one in Figure 12. The maximum – in correspondence of the periapsis – is of $1514\ W/m^2$, while the minimum is $1416\ W/m^2$. This model is not accurate, as we will see in the next paragraph, because of the assumption that the Sun behaves as a black body. In fact, in reality, the maximum is $1399\ W/m^2$ with the minimum being $1309\ W/m^2$ (3). The following one is the code used to plot Figure 11 and Figure 12.

CODE 3---

```
% file name = sun emissive power outside
% date of creation = 01/11/2019
% sun emissive power per unit area , per unit wavelength
clear all
% three parameters of the orbit
A = 6.69*( 10^(-9) );                                    % 1/km
B = 1.12*( 10^(-10) );                                   % 1/km
delta = pi*313/730;
% the two parameters of Plunk's law
C_1 = 3.7415*( 10^(-16) );                               % W*m^2
C_2 = 1.4388*( 10^(-2) );                                % mK
% Stefan-Boltzmann parameter ( W/( (m^2)*(K^4) ) )
SB = 5.670*( 10^(-8) );
% radius of the photosphere (m)
R_S = 696*(10^6);                                        % m
%  temperature of the photosphere (K)
T_S = 5875;
% the array of theta
theta(1) = 0;                                            % winter solstice (21/22 December)
i_ws = 1;
day = 2*pi/365;
days = [1:1:366];
for i = 2:366
  theta(i) = theta(i-1) + day;
  if ( abs( theta(i) - (pi/2) ) <= day )
    i_se = i;                                            % spring equinox (20 March)
  endif
  if ( abs( theta(i) - pi ) <= day )
    i_ss = i;                                            % summer solstice (20/21 June)
  endif
  if ( abs( theta(i) - (3*pi/2) ) <= day )
    i_ae = i;                                            % autumn equinox (22/23 September)
  endif
endfor
% the array of the radius (m)
for i=1:1:366
  o_omega (i) = (10^3)/[ A + ( B*sin(theta(i) + delta ) ) ];   % m
endfor
% the array of the global radiant energy
for i=1:1:366
  J(i) = ( ( R_S/o_omega(i) )^2 )*SB*(T_S)^4;
```

```
endfor
% the coordinates of the orbt with respect to Oxy
for i=1:1:366
  X(i) = o_omega (i)*cos( theta(i) );
  Y(i) = o_omega (i)*sin( theta(i) );
endfor
% the array of lambda in micron for the ultraviolet radiation
lambda_micron (1) = 0.01;
delta_UV = ( 0.4 - 0.01 )/10;
for j= 2:11
  lambda_micron (j) = lambda_micron (j-1) + delta_UV;
endfor
% the array of lambda in micron for the visible spectral region
delta_VS = ( 0.7 - 0.4 )/10;
for j= 12:21
  lambda_micron (j) = lambda_micron (j-1) + delta_VS;
endfor
% the array of lambda in micron for the near infrared spectral region
delta_NI = (3 - 0.7)/10;
for j= 22:31
  lambda_micron (j) = lambda_micron (j-1) + delta_NI;
endfor
% the array of lambda in micron for the intermediate infrared spectral region
delta_II = (25 - 3)/10;
for j= 32:1:41
  lambda_micron (j) = lambda_micron (j-1) + delta_II;
endfor
% the array of lambda in micron for the far infrared spectral region
delta_FI = (1000 - 25)/10;
for j= 42:51
  lambda_micron (j) = lambda_micron (j-1) + delta_FI;
endfor
% the array of lambda in metres
lambda = lambda_micron*( 10^(-6) );
% the array of the emissive power (W/(m^2)*micron)
for i = [i_ws, i_ae, i_ss, i_se]
  for j=1:51
    num = C_1*( (R_S)^2 );
    den = [ (lambda(j)^5)*( (e^(C_2/( lambda(j)*T_S ))) - 1)*( (o_omega(i))^2 ) ]*10^6;
    power(j) = num/den;
  endfor
  % plotting
  figure(i)
  plot (lambda_micron (1:31), power(1:31), '-k', "linewidth", 2)
  xlabel('wavelenght ({\mu})');
  ylabel('W/{\mu}m^{2}');
  axis ([0,3,0,2100]);
  grid minor on
endfor
% plotting
figure(400)
plot ([1:366], J, '-k', "linewidth", 2)
xlabel('day of the year');
ylabel('W/m^{2}');
axis ([0, 366, 1400, 1550])
grid minor on
text (1, 1530, "21/22 Dec.")
```

```
text (i_se, 1530, "20 March")
text (i_ss, 1530, "21 June")
text (i_ae, 1530, "22/23 Sept.")
hold on
plot ([i_se,i_se], [1400,1520], '-k', "linewidth", 1)
plot ([i_ss,i_ss], [1400,1520], '-k', "linewidth", 1)
plot ([i_ae,i_ae], [1400,1520], '-k', "linewidth", 1)
plot ([366,366], [1400,1520], '-k', "linewidth", 1)
------------------------------------------------------------------------------------------------------------------------
```

2.5 The Beer-Bouguer-Lambert law

When solar radiation encounters the atmosphere, there is a loss of energy that is proportional to the length of the path across the medium. This loss is called *extinction* or *attenuation* and it is due mainly to *absorption* and to *scattering* (5). We say that we have absorption when a photon hits an atom's electron with an energy equal to the amount of energy required for the electron to skip from its orbital to the one with higher potential. The result is a loss of energy for the radiant emission (6). The electron that is hit by the electromagnetic field can also oscillate because of it, thus becoming a source of radiation, in turn, whose energy comes from the energy of the electromagnetic field. This phenomenon, called scattering, is another way the solar radiation loses energy when it goes through the atmosphere.

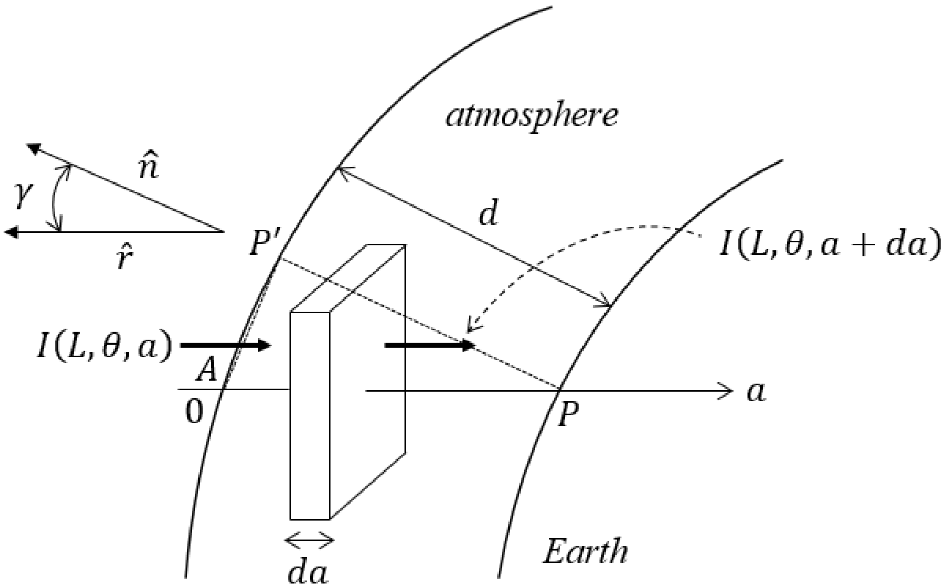


Figure 13. Let the atmosphere have a thickness *d*. Then the distance to be covered for the emission from the sun, in order o reach the soil at sea level is $\boldsymbol{OP = d/\cos\gamma}$.

Infrared radiation is mainly absorbed by O_2, H_2O, CO_2 and N_2O. This absorption happens mainly within the troposphere[1]. On the other hand, the absorption of ultraviolet radiation is due mainly to the layer of Ozone (O_3) which can be found at a height between 20 *km* and 30 *km*. In order to derive a law for the phenomenon of extinction, consider the slice of the atmosphere in Figure 13 (7). With respect to the axis a whose origin is on the outer bound of the atmosphere, we have:

Eq. 41 $$I(L,\theta,a+da) - I(L,\theta,a) = \frac{\partial I(L,\theta,a)}{\partial a} da + o(da)$$

[1] It is the lower layer of the atmosphere. Its upper bound has an average height of about 10 *km*. The stratosphere lies just above the troposphere and its upper bound has a height of about 40 *km* (7).

We make the assumption that $\frac{\partial I(L,\theta,a)}{\partial a}$ is proportional to $I(L,\theta,a)$ according to a parameter $\beta(L) > 0$. Since $\frac{\partial I(L,\theta,a)}{\partial a}$ has to be negative (there is a loss in energy, because of the extinction), we have

Eq. 42 $$\frac{\partial I(L,\theta,a)}{\partial a} = -\beta(L) I(L,\theta,a)$$

This differential equation can be integrated by separation of variables, and we get:

$$\frac{dI}{I} = -\beta(L)da \Rightarrow \int_{I(L,\theta,a=0)}^{I(L,\theta,a)} \frac{dI}{I} = -\beta(L)\int_0^a dt \Rightarrow \ln\frac{I(L,\theta,a)}{I(L,\theta,a=0)} = -\beta(L)a$$

Consider now that $I(L,\theta,a=0)$ is the radiant energy just outside the atmosphere, exactly the value that we calculated in Eq. 38. Hence, we can write:

Eq. 43 $$I(L,\theta,a) = I(L,\theta,a=0)e^{-\beta(L)a} = \frac{C_1 R_S^2}{L^5\left(e^{\frac{C_2}{LT_S}}-1\right)\left(\frac{1}{\frac{6.69}{10^9 km}+\frac{1.12}{10^{10} km}sin\left(\theta+\pi\frac{313}{730}\right)}\right)^2} e^{-\beta(L)a}$$

$L\ [\mu]$	$I(L,m)\ [W/m^2\mu]$					$\beta(L)d$
	$m=0$	$m=1$	$m=4$	$m=7$	$m=10$	
0.15	0.07	0.0	0.0	0.0	0.0	$+\infty$
0.20	10.7	0.0	0.0	0.0	0.0	$+\infty$
0.25	70.4	0.0	0.0	0.0	0.0	$+\infty$
0.30	514	4.1	0.0	0.0	0.0	4.83
0.35	1093	481	40.8	3.5	0.3	0.821
0.40	1429	850	179	37.6	7.9	0.520
0.45	2006	1388	460	153	50.6	0.368
0.50	1942	1451	606	253	106	0.291
0.55	1725	1337	622	289	135	0.255
0.60	1666	1320	656	326	162	0.233
0.65	1511	1257	724	417	240	0.184
0.70	1369	1175	744	471	298	0.153
0.75	1235	1077	713	473	313	0.137
0.80	1109	981	679	470	326	0.123
0.90	891	449	184	92.3	50	0.423
1.00	748	580	354	224	144	0.195
1.50	288	151	88.3	60.2	39.4	0.341
2.00	103	69.9	36.1	17.9	6.5	0.294
5.00	3.79	2.78	1.71	1	0.54	0.224
10.00	0.24	0.0	0.0	0.0	0.0	$+\infty$
1000	0.0	0.0	0.0	0.0	0.0	-

Table 1. Average values of the flux of energy from the sun, at sea level, in function of the air mass *m* and of the wavelength *L* **(3)**. The values for $\boldsymbol{\beta(L)d}$ have been derived from these empirical data, using the Beer-Bouguer-Lambert law.

But we are interested in what happens at sea level, so we have $a = OP = d/\cos\gamma$ – with d being the thickness of the atmosphere – (see Figure 13) and the value of I is:

Eq. 44 $$I\left(L,\theta,a=\frac{d}{\cos\gamma}\right)=\frac{C_1R_S^2}{L^5\left(e^{\frac{C_2}{LT_S}}-1\right)\left(\frac{1}{\frac{6.69}{10^9km}+\frac{1.12}{10^{10}km}sin\left(\theta+\pi\frac{313}{730}\right)}\right)^2}e^{-\frac{\beta(L)d}{\hat{r}\cdot\hat{n}_{noon}}}$$

where we have considered that $\cos\gamma=\hat{r}\cdot\hat{n}_{noon}$. Considering Eq. 7 we can also write

Eq. 45 $$I(L,\theta,\lambda)=\frac{C_1R_S^2e^{-\frac{\beta(L)d}{\cos\lambda\cos\psi_{noon}\sin\theta-(\cos\lambda\sin\psi_{noon}\cos\delta+\sin\lambda\sin\delta)\cos\theta}}}{L^5\left(e^{\frac{C_2}{LT_S}}-1\right)\left(\frac{1}{\frac{6.69}{10^9km}+\frac{1.12}{10^{10}km}sin\left(\theta+\pi\frac{313}{730}\right)}\right)^2}$$

where ψ_{noon} is the value that ψ has at noon. The problem is now to find the function $\beta=\beta(L)$. In order to achieve that goal, we can use a set of empirical data from (3) (see Table 1) and then interpolate them. But before doing that, we have to introduce the definition of *air mass*: it is the ratio between the path covered by sun rays through the atmosphere and the thickness of the atmosphere (8). This quantity is usually indicated *m* and if we consider Figure 13 we have

Eq. 46 $$m=\frac{\overline{AP}}{\overline{PP'}}=\frac{d/\cos\gamma}{d}=\frac{1}{\cos\gamma}$$

which means that

Eq. 47 $$\cos\gamma=\frac{1}{m}\Leftrightarrow\gamma=\cos^{-1}\left(\frac{1}{m}\right)$$

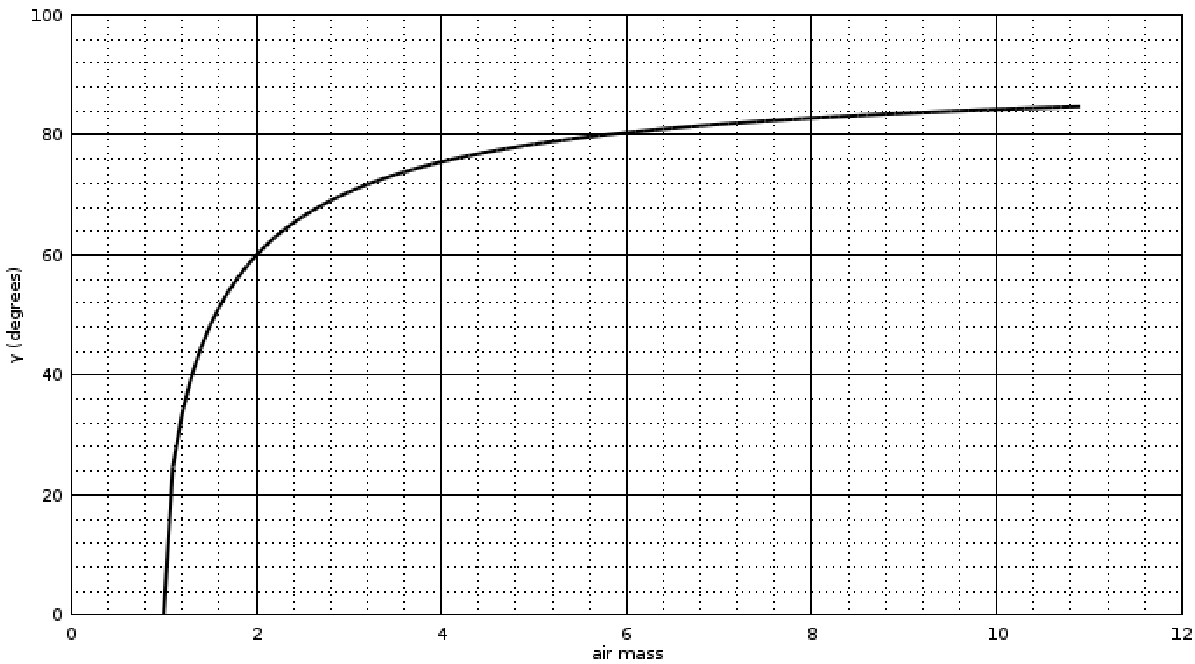


Figure 14. The diagram of the function $\boldsymbol{\gamma}=\boldsymbol{\gamma}(\boldsymbol{m})$.

The diagram of $\gamma=\gamma(m)$ is reported in Figure 14. We can also write:

Eq. 48 $$m=\frac{1}{\cos\lambda\cos\psi_{noon}\sin\theta-(\cos\lambda\sin\psi_{noon}\cos\delta+\sin\lambda\sin\delta)\cos\theta}$$

and we have the *Beer-Bouguer-Lambert law*:

Eq. 49 $$I(L,m) = I(L,m=0)e^{-m\beta(L)d} \Leftrightarrow \beta(L)d = -\frac{1}{m}\ln\frac{I(L,m)}{I(L,m=0)}$$

If we now consider the values of Table 1 corresponding to $L = 5\mu$, we have

$$\ln\frac{I(L,m=1)}{I(L,m=0)} = \ln\frac{2.78}{3.79} = -\beta(5)d \Rightarrow \beta(5)d = 0.31$$

$$\ln\frac{I(L,m=4)}{I(L,m=0)} = \ln\frac{1.71}{3.79} = -4\beta(5)d \Rightarrow \beta(5)d = 0.20$$

$$\ln\frac{I(L,m=7)}{I(L,m=0)} = \ln\frac{1.00}{3.79} = -7\beta(5)d \Rightarrow \beta(5)d = 0.19$$

$$\ln\frac{I(L,m=10)}{I(L,m=0)} = \ln\frac{0.54}{3.79} = -10\beta(5)d \Rightarrow \beta(5)d = 0.19$$

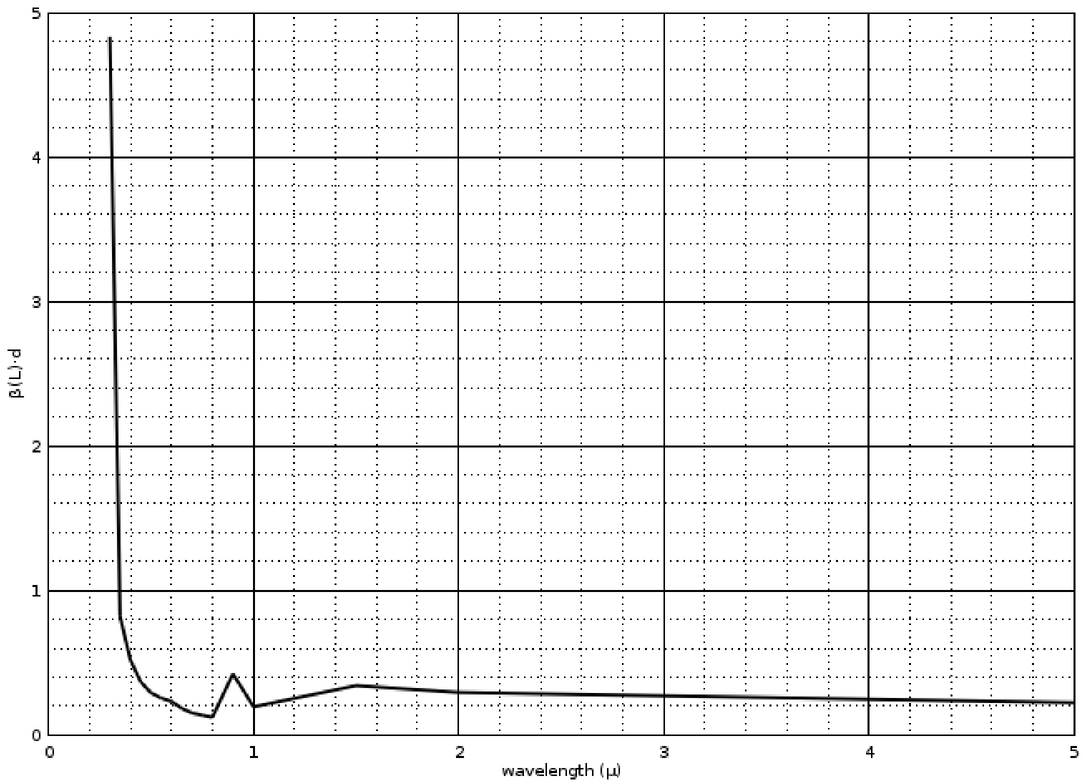


Figure 15. The values of $\beta(L)d$, where $\beta(L)$ is the extinction coefficient in function of the wavelength L and d is the thickness of the atmosphere.

We can then calculate the average value for $\beta(5)d$, which is 0.22. In the same way, we can calculate the other values for $\beta(L)d$ that are reported in the last column on the right of Table 1. The calculation of some of these values is reported in what follows:

$$\beta(2)d = -\frac{\ln\frac{69.9}{103} + \frac{1}{4}\ln\frac{36.1}{103} + \frac{1}{7}\ln\frac{17.9}{103} + \frac{1}{10}\ln\frac{6.50}{103}}{4} = 0.29$$

$$\beta(1.50)d = -\frac{\ln\frac{151}{288} + \frac{1}{4}\ln\frac{88.3}{288} + \frac{1}{7}\ln\frac{60.2}{288} + \frac{1}{10}\ln\frac{39.4}{288}}{4} = 0.34$$

$$\beta(1.00)d = -\frac{\ln\frac{580}{748} + \frac{1}{4}\ln\frac{354}{748} + \frac{1}{7}\ln\frac{224}{748} + \frac{1}{10}\ln\frac{144}{748}}{4} = 0.19$$

$$\beta(0.3)d = -\ln\frac{4.1}{514} = 4.83$$

The values for $\beta(L)d$ in Table 1 are plotted in and have been evaluated by the following code.

CODE 4--

```
% file name = beta_mean
% date of creation = 4/11/2019
clear all
% empirical data
tab (1, 1:6) = [ 0.35, 1093, 481, 40.8, 3.5, 0.3 ];
tab (2, 1:6) = [ 0.40, 1429, 850, 179, 37.6, 7.9 ];
tab (3, 1:6) = [ 0.45, 2006, 1388, 460, 153, 50.6 ];
tab (4, 1:6) = [ 0.50, 1942, 1451, 606, 253, 106 ];
tab (5, 1:6) = [ 0.55, 1725, 1337, 622, 289, 135 ];
tab (6, 1:6) = [ 0.60, 1666, 1320, 656, 326, 162 ];
tab (7, 1:6) = [ 0.65, 1511, 1257, 724, 417, 240 ];
tab (8, 1:6) = [ 0.70, 1369, 1175, 744, 471, 298 ];
tab (9, 1:6) = [ 0.75, 1235, 1077, 713, 473, 313 ];
tab (10, 1:6) = [ 0.80, 1109, 981, 679, 470, 326 ];
tab (11, 1:6) = [ 0.90, 891, 449, 184, 92.3, 50.0 ];
tab (12, 1:6) = [ 1.00, 748, 580, 354, 224, 144 ];
tab (13, 1:6) = [ 1.50, 288, 151, 88.3, 60.2, 39.4 ];
tab (14, 1:6) = [ 2.00, 103, 69.9, 36.1, 17.9, 6.50 ];
tab (15, 1:6) = [ 5.00, 3.79, 2.78, 1.71, 1.00, 0.54 ];
% the array of beta
beta (1,1) = 4.83;
beta (1,2) = 0.30;
for i=1:15
  beta (i+1,1) = -( log( tab(i, 3)/tab(i, 2) ) + ( log( tab(i, 4)/tab(i, 2) )/4 ) + ( log( tab(i, 5)/tab(i, 2) )/7 ) + ( log( tab(i, 6)/tab(i,
2) )/10 )  )/4;
  beta (i+1,2) = tab (i, 1);
endfor
% plotting
plot (beta (1:16, 2), beta (1:16, 1), '-k', "linewidth", 2)
xlabel('wavelength ({\mu})');
ylabel('{\beta}(L){\cdot}d');
grid minor on
save beta.mat beta
save table.mat tab
```

2.6 Monochromatic emissive power at sea level

To calculate the emission of the Sun at sea level, in function of the latitude and of the day of the year, we must use Eq. 44. The problem with that equation is in the extinction coefficient. We only know some values of $\beta(L)d$, for just some wavelengths (see Table 1), while we need a continuous function. One way to solve this problem is to interpolate the values in Table 1, with the method presented in paragraphs 3.2 and 3.2.1. Now that we have the function $\beta(L)d$, we can write a code like the one presented in paragraph 2.4, with just a few changes. In particular, we have to calculate $\hat{r} \cdot \hat{n}_{noon}$, and for that purpose, we can use some lines from the code till_dusk (see paragraph 1.9). The code is the following one and its output for a latitude of $42°$ is the diagram in Figure 16. If we set a latitude of $-32°$, we get the diagram in Figure 17.

CODE 5--

```
% file name = sun emissive power sea level surface
% date of creation = 15/11/2019
% sun emissive power per unit area, per unit wavelength at sea level
clear all
% three parameters of the orbit
A = 6.69*( 10^(-9) );                                                    % 1/km
B = 1.12*( 10^(-10) );                                                   % 1/km
delta = pi*313/730;
% the two parameters of Plunk's law
C_1 = 3.7415*( 10^(-16) );                                               % W*m^2
C_2 = 1.4388*( 10^(-2) );                                                % mK
% Stefan-Boltzmann parameter ( W/( (m^2)*(K^4) ) )
SB = 5.670*( 10^(-8) );
% radius of the photosphere (m)
R_S = 696*(10^6);                                                         % m
%  temperature of the photosphere (K)
T_S = 5875;
% conversion of units of measurments
lambda = 42*pi/180;                                                      % latitude (radiants)
delta = 23.45*pi/180;                                                    % tilt angle
% the array of theta
theta(1) = 0;                          % winter solstice (21/22 December)
i_ws = 1;
day = 2*pi/365;
days = [1:1:366];
for i = 2:366
  theta(i) = theta(i-1) + day;
  if ( abs( theta(i) - (pi/2) ) <= day )
    i_se = i;                          % spring equinox (20 March)
  endif
  if ( abs( theta(i) - pi ) <= day )
    i_ss = i;                          % summer solstice (20/21 June)
  endif
  if ( abs( theta(i) - (3*pi/2) ) <= day )
    i_ae = i;                          % autumn equinox (22/23 September)
  endif
endfor
% the array of the radius (m)
for i=1:1:366
  o_omega (i) = (10^3)/[ A + ( B*sin(theta(i) + delta ) ) ];  % m
endfor
% the array of the wavelength in micron
N = 471;
L(1) = 0.3;
L(N) = 5.0;
delta_L = ( L(N) - L(1) )/(N-1);
```

```
for j = 2:N-1
  L (j) = L(j-1) + delta_L;
endfor
% the array of beta*L
load beta_int.mat S
% the array of L in metres
L_m = L*( 10^(-6) );
% angle psi
psi(1) = 0;
minute = pi/(12*60);
for i = 2:(24*60)+1
  psi(i) = psi(i-1) + minute;
endfor
% angle between n and r at noon
for i= 1:366
  for j=1:(24*60) + 1
    % scalar product between n and r
    scalar_p(j) = [cos(lambda)*sin(psi(j))*cos(delta) + sin(lambda)*sin(delta)]*\
    ( -cos(theta(i)) )+ [(-1)*cos(lambda)*cos(psi(j))]*( -sin(theta(i)) );
  endfor
  % value of psi at noon
  for j=1:(24*60) + 1
    if ( ( scalar_p(j) ) == ( max( scalar_p ) ) )
      j_noon = j;
      psi_noon (i) = psi(j);
    endif
  endfor
  % angle between n and r at noon
  cos_gamma (i) = scalar_p(j_noon);
endfor
% the array of the monochromatic emissive power (W/(m^2)*micron)
for i = 1:366
  for j=1:N
    num = C_1*( (R_S)^2 );
    den = ( (L_m(j)^5)*( (e^(C_2/( L_m(j)*T_S ))) - 1)*( (o_omega(i))^2 ) )*10^6;
    power(j,i) = ( num/den )*( e^(-S(j)/cos_gamma (i)) );
  endfor
endfor
% 2D plotting
figure (1)
plot (L (1:N), power(1:N,i_ws), '-k', "linewidth", 2)
hold on
plot (L (1:N), power(1:N,i_se), '--k', "linewidth", 2)
hold on
plot (L (1:N), power(1:N,i_ss), '-:k', "linewidth", 2)
hold on
plot (L (1:N), power(1:N,i_ae), '-.k', "linewidth", 2)
xlabel('wavelenght ({\mu})');
ylabel('W/{\mu}m^{2}');
axis ([0.3,2,0,1500])
tit = ["Latitude: ", num2str(lambda*180/pi)," degrees"];
title (tit)
legend ('winter solstice','spring equinox','summer solstice', 'autumn equinox', "location",'NORTHEAST')
hold on
plot ([0.4,0.4], [0,1500], '--k', "linewidth", 1)
plot ([0.7,0.7], [0,1500], '--k', "linewidth", 1)
grid on
```

```
% 3D plotting
figure (2)
colormap ([0,0,0])
mesh([1:10:366], L (1:N), power(1:N,1:10:366));
grid on
xlabel('day of the year');
ylabel('wavelenght ({\mu})');
zlabel('W/{\mu}m^{2}');
axis ([1,366,0.3,2,0,1530])
tit = ["Latitude: ", num2str(lambda*180/pi)," degrees"];
title (tit)
```

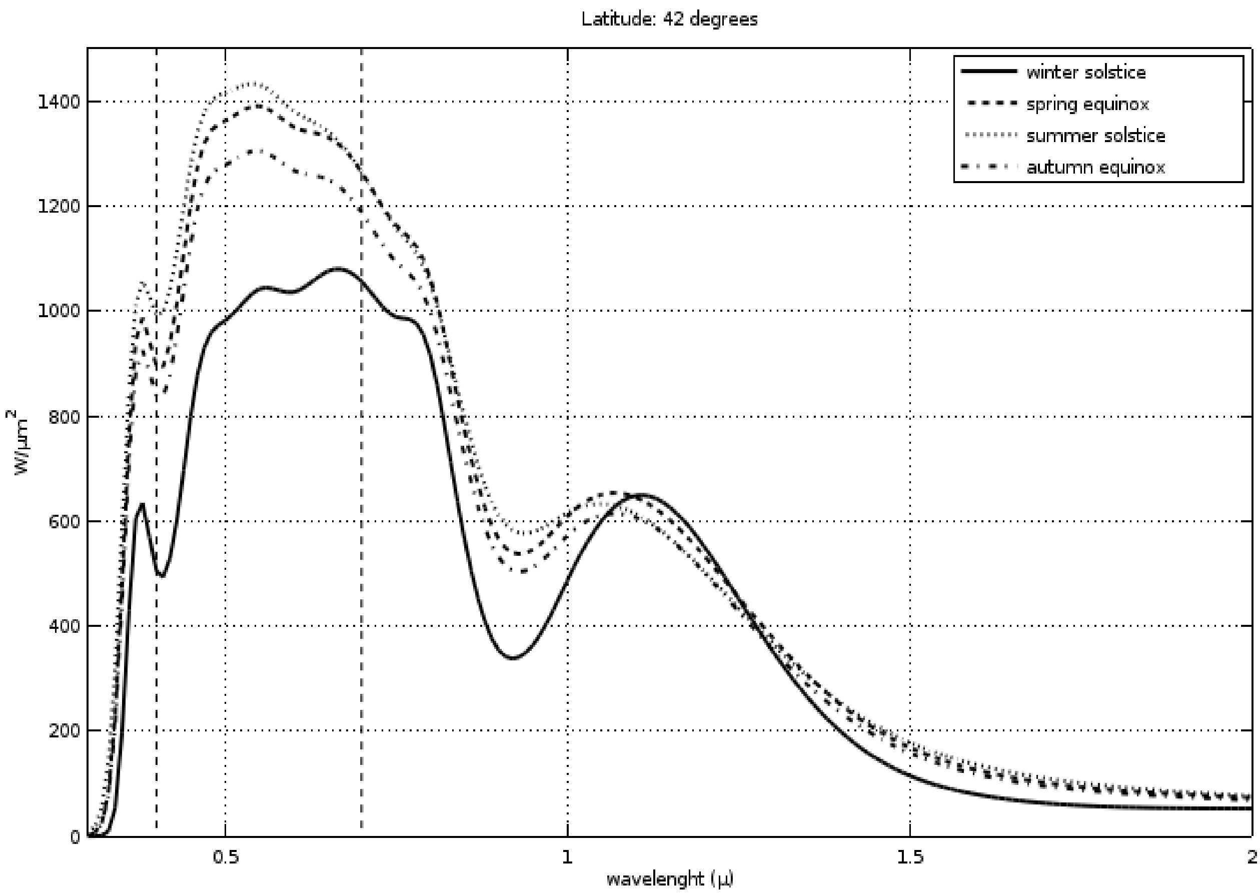


Figure 16. The monochromatic emissive power of the sun, at noon, at sea level, at a latitude of 42° N, in function of the wavelength, for winter solstice, spring equinox, summer solstice and spring equinox.

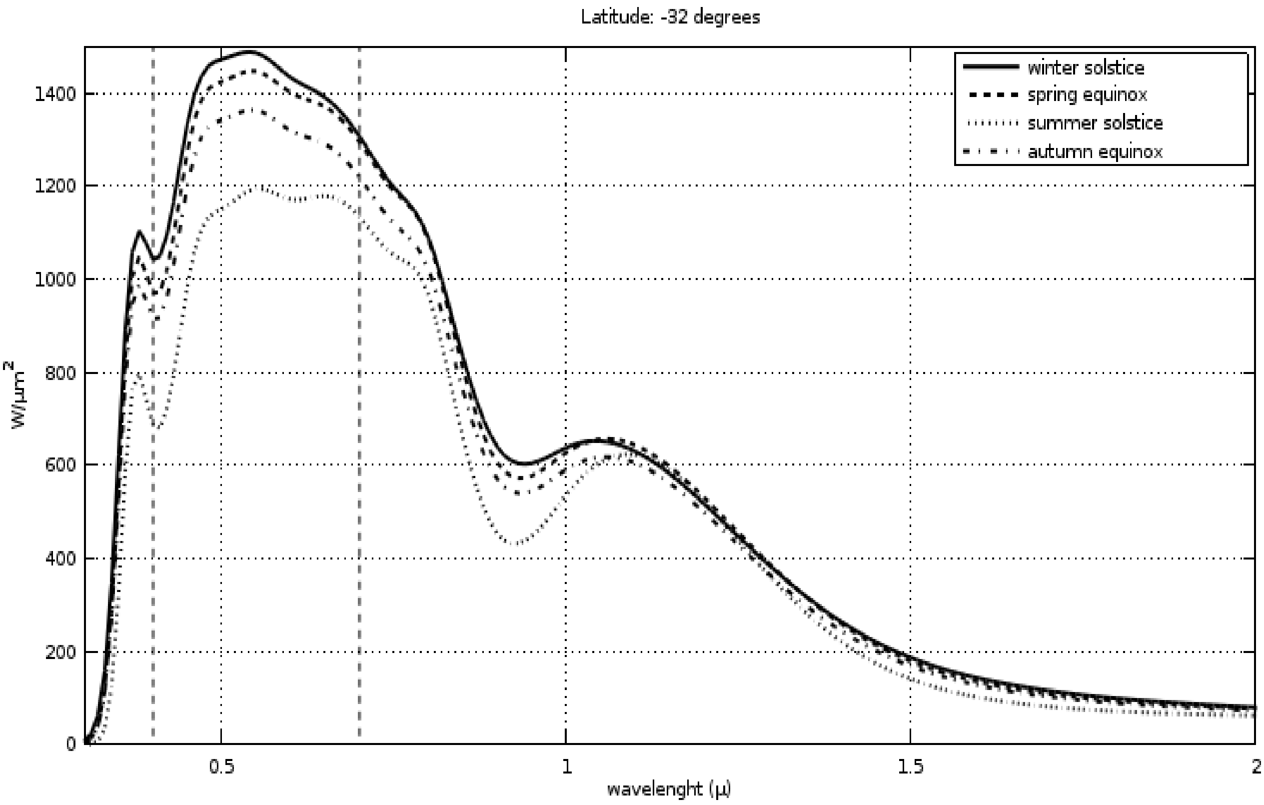


Figure 17. The monochromatic emissive power of the sun, at noon, at sea level, at a latitude of -32° N, in function of the wavelength, for winter solstice, spring equinox, summer solstice and spring equinox.

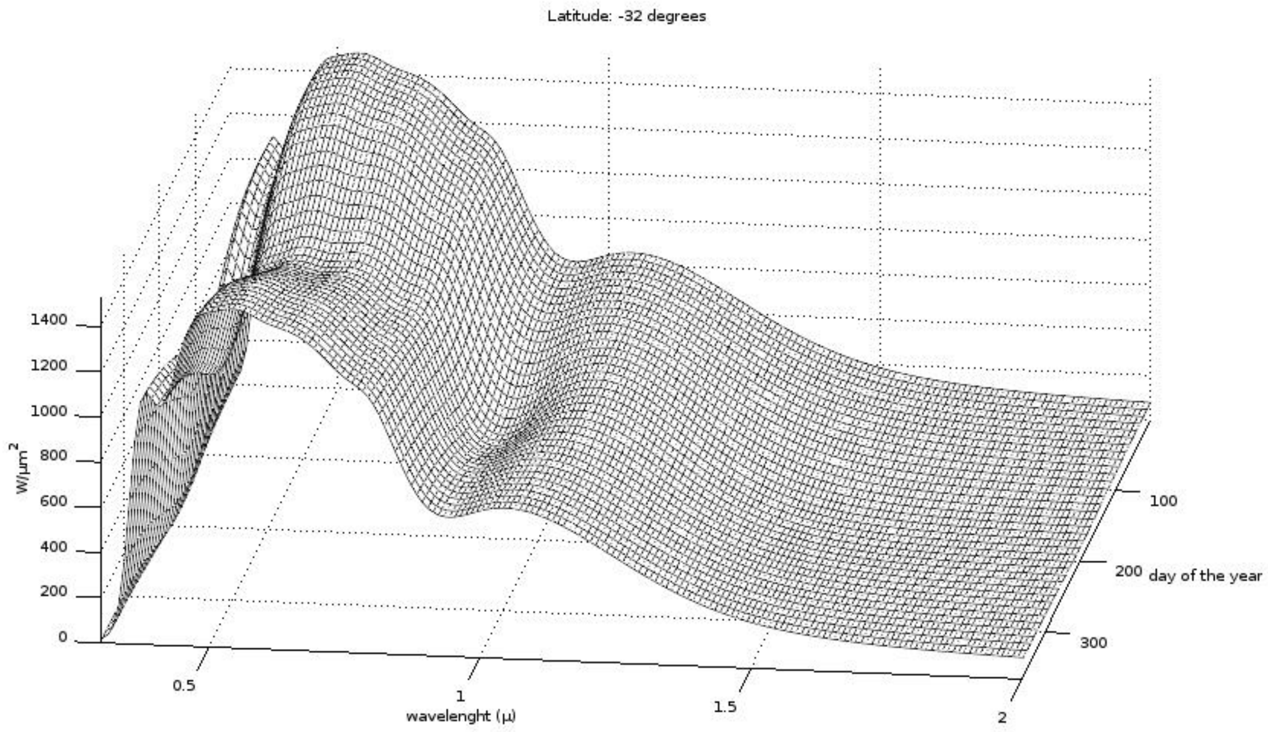


Figure 18. The monochromatic emissive power of the sun at noon, at sea level, at a latitude of -32° N, in function of the wavelength and of the day of the year.

CODE 5 also plots a 3D version of the monochromatic emissive power at sea level, where the emissive power is considered a function of both the wavelength and the day of the year Figure 18.

2.7 Sun emissive power at sea level

For the emissive power at sea level (let's say J_S'), we have to integrate Eq. 44 for L that goes from zero to ∞, analogously at what we did in par. 2.4. We get

Eq. 50 $$J_S'(\theta) = \frac{1}{\left(\frac{1}{\frac{6.69}{10^9 km} + \frac{1.12}{10^{10} km} sin\left(\theta + \pi \frac{313}{730}\right)}\right)^2} \int_0^\infty \frac{C_1 R_S^2 e^{-\frac{\beta(L)d}{\hat{r}\cdot\hat{n}_{noon}}}}{L^5\left(e^{\frac{C_2}{LT_S}} - 1\right)} dL$$

As the reader might have noticed, we can't now use Eq. 63 (as we did in par. 2.4) because we have now the function of extinction inside the integral. But if we consider the interpolation of $\beta(L)d$ proposed in par. 3.2.1, we can write

Eq. 51 $$\int_{L_1}^{L_n} \frac{C_1 R_S^2 e^{-\frac{S(L)}{\hat{r}\cdot\hat{n}_{noon}}}}{L^5\left(e^{\frac{C_2}{LT_S}} - 1\right)} dL = \sum_{i=1}^{n-1} \int_{L_i}^{L_{i+1}} \frac{C_1 R_S^2 e^{-\frac{s_i(L)}{\hat{r}\cdot\hat{n}_{noon}}}}{L^5\left(e^{\frac{C_2}{LT_S}} - 1\right)} dL$$

Now, if we consider Eq. 90, we have

Eq. 52 $$\int_{L_1}^{L_n} \frac{C_1 R_S^2 e^{-\frac{S(L)}{\hat{r}\cdot\hat{n}_{noon}}}}{L^5\left(e^{\frac{C_2}{LT_S}} - 1\right)} dL = \sum_{i=1}^{n-1} \int_{L_i}^{L_{i+1}} \frac{C_1 R_S^2 e^{-\frac{\frac{M_{i+1}(L-L_i)^3 + M_i(L_{i+1}-L)^3}{6(L_{i+1}-L_i)} + C_i L + D_i}{\hat{r}\cdot\hat{n}_{noon}}}}{L^5\left(e^{\frac{C_2}{LT_S}} - 1\right)} dL$$

For the integral within the sum, we can write

$$C_1 R_S^2 e^{-\frac{D_i}{\hat{r}\cdot\hat{n}_{noon}}} \int_{L_i}^{L_{i+1}} \frac{e^{-\frac{\frac{M_{i+1}(L-L_i)^3 + M_i(L_{i+1}-L)^3}{6(L_{i+1}-L_i)} + C_i L}{\hat{r}\cdot\hat{n}_{noon}}}}{L^5\left(e^{\frac{C_2}{LT_S}} - 1\right)} dL$$

and with a few passages, we get

$$C_1 R_S^2 e^{\frac{M_i L_{i+1}^3 - M_{i+1} L_i^3 - 6(L_{i+1}-L_i) D_i}{6\hat{r}\cdot\hat{n}_{noon}(L_{i+1}-L_i)}} \int_{L_i}^{L_{i+1}} \frac{e^{-\frac{(M_{i+1}-M_i)L^3 - 3(M_{i+1}L_i - M_i L_{i+1})L^2 + \left[3\left(M_{i+1}L_i^2 - M_i L_{i+1}^2\right) + 6C_i(L_{i+1}-L_i)\right]L}{6\hat{r}\cdot\hat{n}_{noon}(L_{i+1}-L_i)}}}{L^5\left(e^{\frac{C_2}{LT_S}} - 1\right)} dL$$

This is an integral of the following type:

Eq. 53 $$\int \frac{e^{ax^3+bx^2+cx}}{x^5\left(e^{\frac{h}{x}} - 1\right)}$$

I have tried to solve it analytically, without success (but it might still very well have an analytical solution). So, the only available avenue is to solve the integral numerically. We can use the expression of $\beta(L)d$ calculated by interpolation in par. 3.2.1 and apply Simpson's method of integration. Given the function $f: [a, c] \rightarrow \mathbb{R}$, Simpson's method states that its integral in $[a, c]$ can be approximated with the integral below the parabola in Figure 19. Hence, we can write:

Eq. 54 $$\int_a^c f(x)dx \cong (c-a)\frac{f(a)+4f(b)+f(c)}{3}$$

While writing a code, if we name int_n the value of the area for the n^{th} iterate and if f_n is the value of f in x_n, we have

Eq. 55 $$int_{n+2} = int_n + 2\Delta_x \frac{f_n+4f_{n+1}+f_{n+2}}{3}$$

where we have assumed that $\Delta_x = x_{n+1} - x_n$ for any value of n (9).

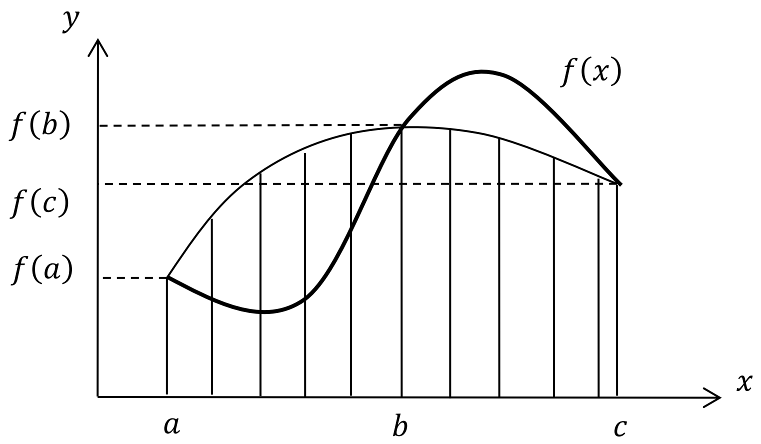


Figure 19. Simpson's method. The integral of $f: [a, c]$ is approximated with the area below the parabola for the three points: $[a, f(a)]$, $[b, f(b)]$ and $[c, f(c)]$.

That said, we can easily write the following code, which uses the values of $\beta(L)d$ calculated by CODE 7. It numerically integrates the interpolated function $S(x)$ within each of the intervals $[x_i, x_{i+1}]$. With the symbolism adopted in par. 3.2 and considering the numeric method in Eq. 54, we have

$$\int_{L_i}^{L_{i+1}} f_i(L)dL \cong 2\Delta_i \frac{f_i(L_i) + 4f_i(L_i + \Delta_i) + f_i(x_i + 2\Delta_i)}{3} +$$

$$+2\Delta_i \frac{s_i(L_i + 2\Delta_i) + 4s_i(L_i + 3\Delta_i) + s_i(L_i + 4\Delta_i)}{3} +$$

$$+2\Delta_i \frac{f_i(f_i + 4\Delta_i) + 4f_i(f_i + 5\Delta_i) + f_i(L_i + 6\Delta_i)}{3} + \cdots \Rightarrow$$

Eq. 56 $$\int_{L_i}^{L_{i+1}} s_i(L)dx \cong$$

$$= \frac{2\Delta_i}{3}\sum_{k=1}^{m}[f_i(L_i + (2k-2)\Delta_i) + 4f_i(L_i + (2k-1)\Delta_i) + f_i(L_i + 2k\Delta_i)]$$

with $\Delta_i = \frac{L_{i+1}-L_i}{m-1}$ and m and odd number (it must be odd, to use Simpson's algorithm). Then we have:

Eq. 57 $$J'_S(\theta) = \frac{\sum_{i=1}^{n-1}\frac{2\Delta_i}{3}\sum_{k=1}^{m}[f_i(L_i+(2k-2)\Delta_i)+4f_i(L_i+(2k-1)\Delta_i)+f_i(L_i+2k\Delta_i)]}{\left(\frac{1}{\frac{6.69}{10^9 km}+\frac{1.12}{10^{10} km}sin\left(\theta+\pi\frac{313}{730}\right)}\right)^2}$$

The code that calculates $J'_S(\theta)$ uses almost all the lines of CODE 5, which calculates the functions $f_1, f_2, \dots, f_{n-1}$. Then it performs the numerical integration. The output for a latitude of 42° north can be found in Figure 20.

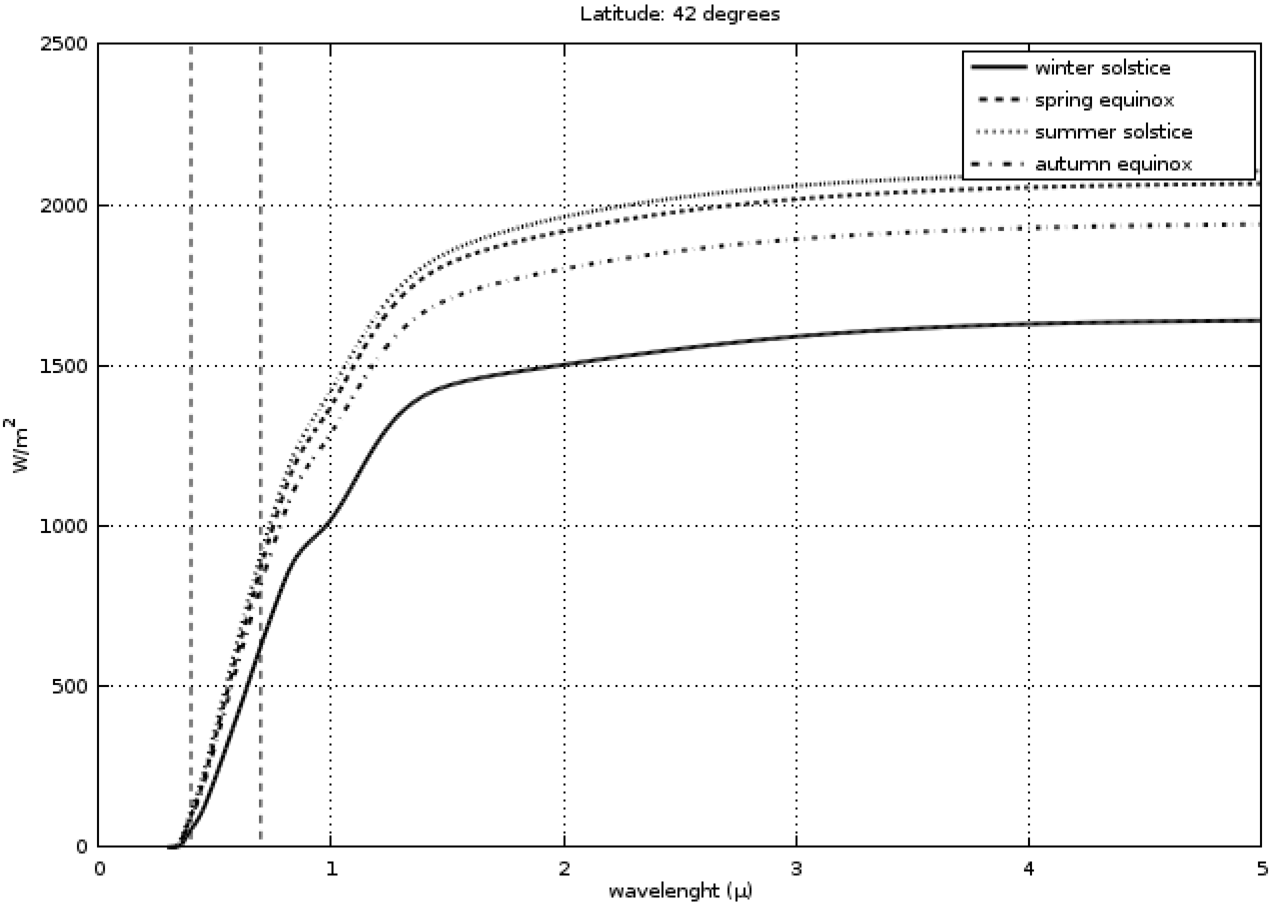


Figure 20. Sun emissive power at sea level at a latitude of 42 degrees north.

CODE 6---

```
% file name = sun emissive power sea level_integration
% date of creation = 07/11/2019
% sun emissive power per unit area, per unit wavelength at sea level
clear all
% three parameters of the orbit
A = 6.69*( 10^(-9) );                                                     % 1/km
B = 1.12*( 10^(-10) );                                                    % 1/km
```

```
delta = pi*313/730;
% the two parameters of Plunk's law
C_1 = 3.7415*( 10^(-16) );                                      % W*m^2
C_2 = 1.4388*( 10^(-2) );                                       % mK
% Stefan-Boltzmann parameter ( W/( (m^2)*(K^4) ) )
SB = 5.670*( 10^(-8) );
% radius of the photosphere (m)
R_S = 696*(10^6);                                               % m
%  temperature of the photosphere (K)
T_S = 5875;
% conversion of units of measurments
lambda = 42*pi/180;                                             % latitude (radiants)
delta = 23.45*pi/180;                                           % tilt angle
% the array of theta
theta(1) = 0;                          % winter solstice (21/22 December)
i_ws = 1;
day = 2*pi/365;
days = [1:1:366];
for i = 2:366
  theta(i) = theta(i-1) + day;
  if ( abs( theta(i) - (pi/2) ) <= day )
    i_se = i;                          % spring equinox (20 March)
  endif
  if ( abs( theta(i) - pi ) <= day )
    i_ss = i;                          % summer solstice (20/21 June)
  endif
  if ( abs( theta(i) - (3*pi/2) ) <= day )
    i_ae = i;                          % autumn equinox (22/23 September)
  endif
endfor
% the array of the radius (m)
for i=1:1:366
  o_omega (i) = (10^3)/[ A + ( B*sin(theta(i) + delta ) ) ];     % m
endfor
% the array of the wavelength in micron
N = 471;
L(1) = 0.3;
L(N) = 5.0;
delta_L = ( L(N) - L(1) )/(N-1);
for j = 2:N-1
  L (j) = L(j-1) + delta_L;
endfor
% the array of beta*L
load beta_int.mat S
% the array of L in metres
L_m = L*( 10^(-6) );
delta_L_m = delta_L*( 10^(-6) );
% angle psi
psi(1) = 0;
minute = pi/(12*60);
for i = 2:(24*60)+1
  psi(i) = psi(i-1) + minute;
endfor
% angle between n and r at noon
for i= [i_ws, i_ae, i_ss, i_se]
  for j=1:(24*60) + 1
    % scalar product between n and r
```

```
    scalar_p(j) = [cos(lambda)*sin(psi(j))*cos(delta) + sin(lambda)*sin(delta)]*\
    ( -cos(theta(i)) )+ [(-1)*cos(lambda)*cos(psi(j))]*( -sin(theta(i)) );
  endfor
  % value of psi at noon
  for j=1:(24*60) + 1
    if ( ( scalar_p(j) ) == ( max( scalar_p ) ) )
      j_noon = j;
      psi_noon (i) = psi(j);
    endif
  endfor
  % angle between n and r at noon
  cos_gamma (i) = scalar_p(j_noon);
endfor
% the array of the emissive power ( W/(m^2) )
for i = [i_ws, i_ae, i_ss, i_se]
  for j=1:N
    num = C_1*( (R_S)^2 );
    den = ( (L_m(j)^5)*( (e^(C_2/( L_m(j)*T_S ))) - 1)*( (o_omega(i))^2 ) )*10^6;
    power(j,i) = ( num/den )*( e^(-S(j)/cos_gamma (i)) );
  endfor
  % integration
  power_int (1,i) = 0.;
  power_int (2,i) = power(2,i)*delta_L/2.;
  for k=1:N-2
    power_int (k+2,i) = power_int (k,i) + 2*delta_L*( power(k,i) + ( 4*power(k+1,i) ) + power(k+2,i) )/3.;
  endfor
  % plotting
  figure(i)
  plot (L (1:N), power_int (1:N,i), '-k', "linewidth", 2)
  xlabel('wavelenght ({\mu})');
  ylabel('W/m^{2}');
  tit = ["Latitude: ", num2str(lambda*180/pi)," degrees"];
  title (tit)
  grid minor on
  if (i == i_ws)
    legend ('winter solstice', "location",'NORTHEAST')
  endif
  if (i == i_se)
    legend ('spring equinox', "location",'NORTHEAST')
  endif
  if (i == i_ss)
    legend ('summer solstice', "location",'NORTHEAST')
  endif
  if (i == i_ae)
    legend ('autumn equinox', "location",'NORTHEAST')
  endif
  hold on
  plot ([0.4,0.4], [0,2500], '--k', "linewidth", 1)
  plot ([0.7,0.7], [0,2500], '--k', "linewidth", 1)
endfor
% plotting
figure (400)
plot (L (1:N), power_int (1:N,i_ws), '-k', "linewidth", 2)
hold on
plot (L (1:N), power_int (1:N,i_se), '--k', "linewidth", 2)
hold on
plot (L (1:N), power_int (1:N,i_ss), '-:k', "linewidth", 2)
```

```
hold on
plot (L (1:N), power_int (1:N,i_ae), '-.k', "linewidth", 2)
xlabel('wavelenght ({\mu})');
ylabel('W/m^{2}');
tit = ["Latitude: ", num2str(lambda*180/pi)," degrees"];
title (tit)
legend ('winter solstice','spring equinox','summer solstice', 'autumn equinox', "location",'NORTHEAST')
hold on
plot ([0.4,0.4], [0,2500], '--k', "linewidth", 1)
plot ([0.7,0.7], [0,2500], '--k', "linewidth", 1)
grid on
```

----

3 Mathematical notes

3.1 Stefan-Boltzmann law

The so-called Stefan-Boltzmann law is the calculation of the improper integral in Eq. 39. Let's consider the substitution:

Eq. 58 $$x = e^{\frac{C_2}{LT}} \Rightarrow L = \frac{C_2}{T\ln x}, dL = -\frac{C_2}{xT\ln^2 x}dx$$

Then we have

Eq. 59 $$\int_0^\infty \frac{C_1}{L^5\left(e^{\frac{C_2}{LT}S}-1\right)}dL = \frac{T^4 C_1}{C_2^4}\int_1^\infty \frac{\ln^3 x}{x(x-1)}dx$$

Let's now consider the further substitution

Eq. 60 $$h = \frac{1}{x} \Rightarrow x = \frac{1}{h}, dx = -\frac{1}{h^2}dh$$

We have the integral

Eq. 61 $$\int_0^\infty \frac{C_1}{L^5\left(e^{\frac{C_2}{LT}S}-1\right)}dL = \frac{T^4 C_1}{C_2^4}\int_1^0 \frac{\ln^3 h}{1-h}dh$$

F. Alg. rat. fract. à dén. binôme; Log. en num. $(lx)^a$ pour a spécial. TABLE 109, suite. Lim. 0 et 1.

5) $\int (lx)^2 \frac{1+x^2}{1+x^4}dx = \frac{3}{64}\pi^3\sqrt{2}$ (VIII, 568). 6) $\int (lx)^2 \frac{1-x^4}{1-x^6}dx = \frac{1}{27}\pi^3\sqrt{3}$ (IV, 219).

7) $\int (lx)^2 \frac{x^{p-q-1}+x^{p+q-1}}{1+x^{2p}}dx = \frac{\pi^3}{8p^3}\left(2\,Sec^3\frac{q\pi}{2p} - Sec\frac{q\pi}{2p}\right)$ (VIII, 568).

8) $\int (lx)^2 \frac{x^{p-q-1}-x^{p+q-1}}{1-x^{2p}}dx = \frac{\pi^3}{4p^3}\,Sin\frac{q\pi}{2p}\,.\,Sec^3\frac{q\pi}{2p}$ (VIII, 568).

9) $\int (lx)^3 \frac{dx}{1+x} = -\frac{7}{120}\pi^4$ (IV, 220).

10) $\int (lx)^3 . x^a \frac{dx}{1+x} = (-1)^{a-1}\sum_a^\infty \frac{(-1)^n}{(n+1)^4}$ Oettinger, Gr. 39, 425.

11) $\int (lx)^3 \frac{dx}{1-x} = -\frac{1}{15}\pi^4$ (IV, 220).

Figure 21. A table from the 1867 edition of Nouvelles Tables d'Intégrales Definiés. Equation 11 is the integral in Eq. 61 of the present book.

This integral can be found here (10), equation 11 of table 109, page 157 (see Figure 21). The same integral is reported in (11), on page 541. The solution is:

Eq. 62 $\int_1^0 \frac{\ln^3 h}{1-h} dh = \frac{\pi^4}{15} \sim 6.49394 \Rightarrow$

Eq. 63 $\int_0^\infty \frac{C_1}{L^5\left(e^{\frac{C_2}{LT}S}-1\right)} dL = \frac{T^4 C_1}{15 C_2^4} \pi^4 \sim 5.6696 \cdot 10^{-8} T^4 \frac{W}{m^2 K^4}$

3.1.1 Integration by series

Sadly, these two tables of integrals mentioned, do not provide the method they used to find the solution in Eq. 62. I will now provide a way to get that result. Let's consider the series:

Eq. 64 $1 + h + h^2 + \cdots + h^k + \cdots = \sum_{k=0}^\infty h^k$

This is the well-known geometric series, which converges to $\frac{1}{1-h}$ for $-1 < h < 1$. It not only converges; it is totally convergent within $]-1,1[$ (its radius of convergence is $\rho = 1$) and this is easily proved by application of the criteria by d'Alembert (12). So, we have

Eq. 65 $\frac{1}{1-h} = \sum_{k=0}^\infty h^k, h \in]-1,1[\Rightarrow$

Eq. 66 $\frac{\ln^3 h}{1-h} = \ln^3 h \sum_{k=0}^\infty h^k, h \in]0,1[$

We can now integrate Eq. 61 by series:

Eq. 67 $\int_1^0 \frac{\ln^3 h}{1-h} dh = \sum_{k=0}^\infty \int_1^0 h^k \ln^3 h \, dh$

Eq. 68 $\int_0^\infty \frac{C_1}{L^5\left(e^{\frac{C_2}{LT}S}-1\right)} dL = \frac{T^4 C_1}{C_2^4} \sum_{k=0}^\infty \int_1^0 h^k \ln^3 h \, dh$

If we now consider the substitution $x^2 = -\ln h \Rightarrow h = e^{-x^2}, dh = -2xe^{-x^2} dx$, we have

Eq. 69 $\int_1^0 h^k \ln^3 h \, dh = 2 \int_0^\infty x^7 e^{-(k+1)x^2} dx$

The reader might have recognized that we are now dealing with one of the integrals from the *error theory* (13). Consider the following integral:

$$I(n,\lambda) = \int_0^{+\infty} x^n e^{-\lambda x^2} dx = -\frac{1}{2\lambda} \int_0^{+\infty} x^{(n-1)} d\left(e^{-\lambda x^2}\right) = -\frac{1}{2\lambda}\left(\left(x^{(n-1)} e^{-\lambda x^2}\right)_{x=0}^{x=+\infty} - \int_0^{+\infty} e^{-\lambda x^2} dx^{(n-1)}\right) =$$

$$= -\frac{1}{2\lambda}\left(0 - (n-1)\int_0^{+\infty} e^{-\lambda x^2} x^{(n-2)} dx\right) = \frac{n-1}{2\lambda} I(n-2,\lambda)$$

When n is odd, we have

$$I(n,\lambda) \triangleq \int_0^{+\infty} x^n e^{-\lambda x^2} dx = \frac{n-1}{2\lambda}\frac{n-3}{2\lambda}\frac{n-5}{2\lambda} \ldots \frac{n-n}{2\lambda} I(n-n+1,\lambda) = \frac{(n-1)!!}{(2\lambda)^{\frac{n-1}{2}}} I(1,\lambda)$$

But we have

$$I(1,\lambda) = \int_0^{+\infty} x e^{-\lambda x^2} dx = \int_0^{+\infty} e^{-\lambda x^2} dx^2 = -\frac{1}{2\lambda}\int_0^{+\infty} e^{-\lambda x^2} d(-\lambda x^2) = -\frac{1}{2\lambda} e^{-\lambda x^2}\Big|_0^{+\infty} = \frac{1}{2\lambda}$$

So, we get

Eq. 70 $\qquad \int_0^{+\infty} x^n e^{-\lambda x^2} dx = \frac{(n-1)!!}{(2\lambda)^{\frac{n-1}{2}}}\frac{1}{2\lambda},$ $\qquad$ when n is odd

which means that for $n = 7$ and $\lambda = k + 1$ we can write

Eq. 71 $\qquad \int_0^{\infty} x^7 e^{-(k+1)x^2} dx = \frac{6!!}{\left(2(k+1)\right)^4} = \frac{3}{(k+1)^4}$

Thus, by substituting Eq. 71 in Eq. 69 we have

Eq. 72 $\qquad \int_1^0 h^k \ln^3 h \, dh = \frac{6}{(k+1)^4}$

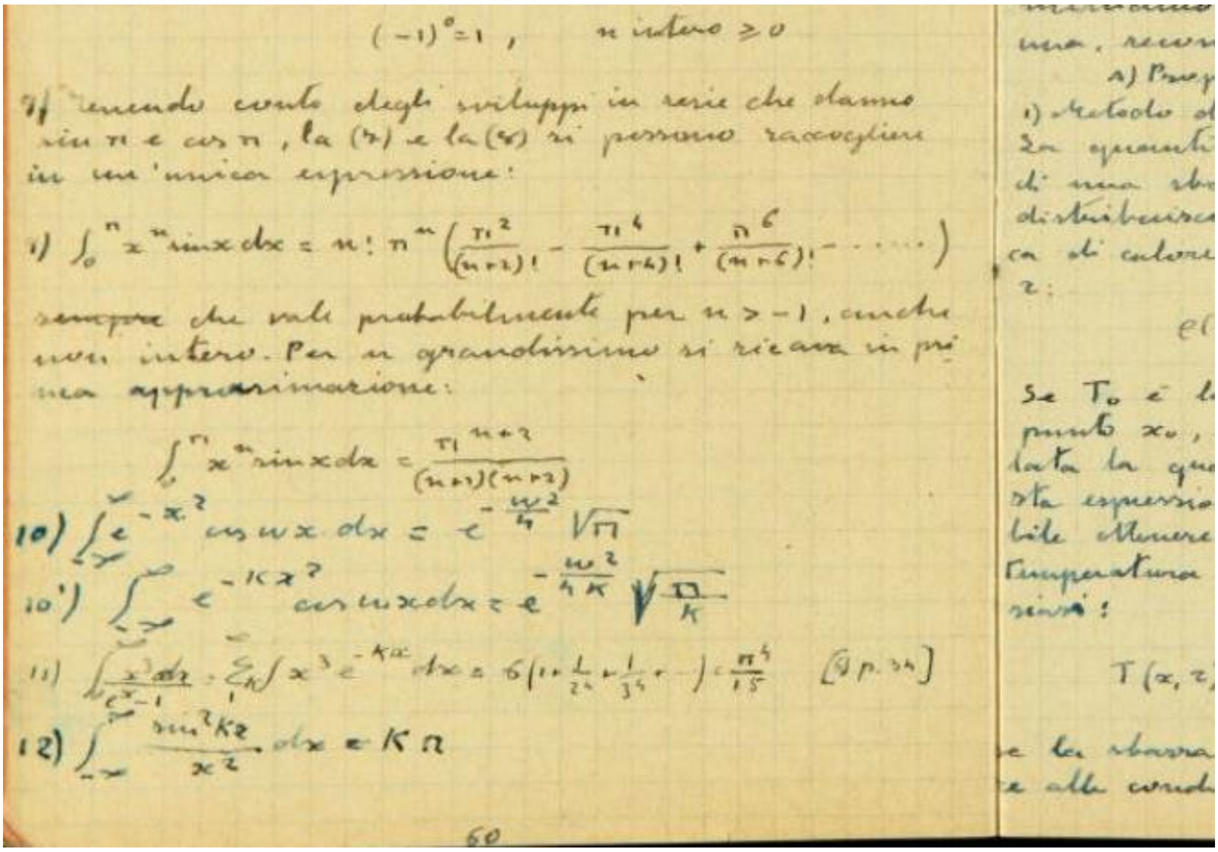


Figure 22. The exact solution of the improper integral of the Stefan-Boltzmann law (eq. 11), in the notes by a young Ettore Majorana, when he was a student (about 1927, he was 21 or so). He wrote a reference to a previous page of his notes (on the right), where he collected several numerical series, the Reimann zeta function for 4, among them.

By substituting Eq. 72 in Eq. 67 we get

Eq. 73 $\qquad \int_1^0 \frac{\ln^3 h}{1-h} dh = 6\sum_{k=1}^{\infty}\frac{1}{k^4} \sim 6.493937$

where I have used the first 100 addends of the series (a generalized harmonic series (14)) and I have truncated the result at the 7th digit. We get a pretty decent approximation of the exact result in Eq. 62. The exact solution requires some considerations about the generalized harmonic series, more precisely about the *hyperharmonic series* (see below). The substitution of Eq. 69 in Eq. 68 leads to the expression:

Eq. 74 $$\int_0^{\infty} \frac{C_1}{L^5\left(e^{\frac{C_2}{LT}S}-1\right)} dL = 2\frac{T^4C_1}{C_2^4}\sum_{k=0}^{\infty}\int_0^{\infty} x^7 e^{-(k+1)x^2} dx = 6\frac{T^4C_1}{C_2^4}\sum_{k=1}^{\infty}\frac{1}{k^4}$$

where we have also considered Eq. 71. This solution is the same one that we find in the notes by the Italian physicist Ettore Majorana, written when he was a 21 years old student of engineering, around the year 1927 (see Figure 22). In particular, the equation 1.371 of his notes, recently translated into English by a group of Italian physicists (15), says:

Eq. 75 $$\int_{-\infty}^{+\infty} \frac{y^3}{e^y-1} dy = \sum_{k=1}^{\infty}\int_{-\infty}^{\infty} y^3 e^{-ky} dy = 6\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right) = \frac{\pi^4}{15}$$

Now, at a first look, it might seem that this integral is different from the one of the Stefan-Boltzmann law, but if we consider the substitution $y = L^{-1}$ in the left hand of Eq. 75 and the substitution $y = x^2$ in the integrals within the sum we get a more recognizable expression:

Eq. 76 $$\int_0^{+\infty} \frac{dL}{L^5\left(e^{\frac{1}{L}}-1\right)} = \sum_{k=1}^{\infty}\int_0^{\infty} x^7 e^{-kx^2} dx = 6\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right) = \frac{\pi^4}{15}$$

It is interesting to note that Majorana used the expressions of the integrals with the smaller exponents.

3.1.2 Hyperharmonic series

The solution of Eq. 73 requires the calculation of the sum $\sum_{h=1}^{\infty}\frac{1}{h^4}$, which is an example of hyperharmonic series, whose general case is

Eq. 77 $$\zeta(p) = \sum_{h=1}^{\infty}\frac{1}{h^p}, p = 2, 3, 4 \ldots$$

The function $\zeta = \zeta(x), x \in \mathbb{R}$ is named *Riemann zeta function* and we know from the work by Leonhard Euler (1707-1783) its values for $x = 2n$:

Eq. 78 $$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|, \quad n = 1, 2, \ldots$$

where B_n are the so-called Bernoulli's coefficients, and they are defined by the following equation (16):

Eq. 79 $$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

I would like to say now that we won't prove Eq. 87 because "this is beyond the scope of the present book", but to be honest the reality is that I have no idea how Euler derived Eq. 87, so even if we needed to prove the general formula, I would not be able to deliver the proof. Luckily, in order to calculate the right hand of Eq. 74 we just need to calculate $\zeta(4)$. Consider the Fourier series (17) of $f: x \mapsto x^2$, defined in $[-\pi, \pi]$:

Eq. 80 $$x^2 = a_0 + \sum_{k=1}^{\infty}[a_k \cos(kx) + b_k \sin(kx)]$$

For the first coefficient, we have $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{\pi^2}{3}$. The coefficients $b_1, b_2, \ldots$ are zero, since the integrand function is symmetric with respect to $x = 0$. Moreover, we have:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(kx)\, dx = \frac{2}{\pi k}\int_0^{\pi} x^2 d(\sin(kx)) = \frac{2}{\pi k}\left\{[x^2 \sin(kx)]_0^{\pi} - \int_0^{\pi} \sin(kx)\, d(x^2)\right\} =$$

$$= -\frac{4}{\pi k}\int_0^{\pi} x \sin(kx)\, dx = \frac{4}{\pi k^2}\int_0^{\pi} x d[\cos(kx)] = \frac{4}{\pi k^2}\left\{[x \cos(kx)]_0^{\pi} - \int_0^{\pi} \cos(kx)\, dx\right\} =$$

$$= \frac{4}{\pi k^2}\left\{\pi \cos(k\pi) - \frac{1}{k}[\sin(kx)]_0^{\pi}\right\} = \frac{4\cos(k\pi)}{k^2} = \frac{4(-1)^k}{k^2}$$

Hence, the Fourier series for f is

Eq. 81 $$x^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}\left[\frac{(-1)^k \cos(kx)}{k^2}\right]$$

Now, since Parseval's equality says that

Eq. 82 $$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = 2a_0^2 + \sum_{k=1}^{\infty}[a_k^2 + b_k^2]$$

we can write

$$\frac{2}{\pi}\int_0^{\pi} x^4 dx = 2\frac{\pi^4}{9} + \sum_{k=1}^{\infty}\left(\frac{4(-1)^k}{k^2}\right)^2 \Rightarrow 2\frac{\pi^4}{5} = 2\frac{\pi^4}{9} + 16\sum_{k=1}^{\infty}\frac{1}{k^4} \Rightarrow$$

Eq. 83 $$\sum_{k=1}^{\infty}\left(\frac{1}{k^4}\right) = \frac{\pi^4}{90}$$

which is the value $\zeta(4)$ we were searching for. By substituting it in Eq. 74, the solution of the improper integral called Stefan-Boltzmann law is completed.

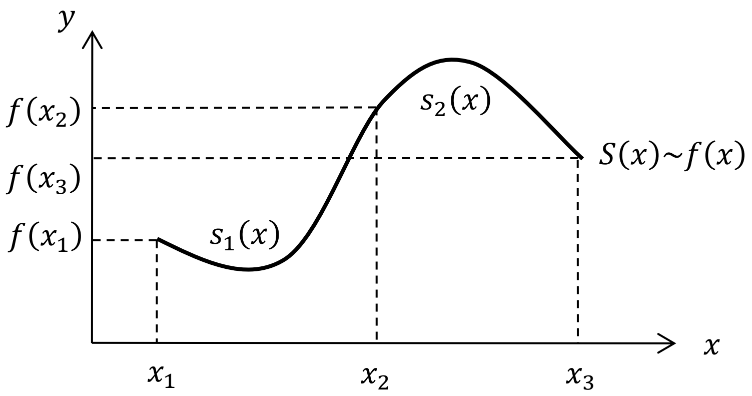
3.2 Interpolation

Let's suppose that we know a set of data (x_i, f_i), $i = 1, 2, \ldots, n$ where f_i is the value assumed by the function $f(x)$ for $x = x_i$, with $x_i \leq x_{i+1}$. We want to find an expression for the function $f(x)$. One way to do that is to build a function $S: [x_1, x_n] \to \mathbb{R}$ as a set of polynomials $s_i: [x_i, x_{i+1}] \to \mathbb{R}$ of the third order (for instance) so that the following conditions are satisfied:

Eq. 84 $$\begin{cases} S(x_1) = s_1(x_1) \\ S(x_i) = s_{i-1}(x_i) = s_i(x_i) = f_i \quad i = 2, \ldots, n-1 \\ S(x_n) = s_n(x_n) \end{cases}$$

Eq. 85 $$s'_{i-1}(x_i) = s'_i(x_i) \qquad i = 2, \dots, n-1$$

Eq. 86 $$s''_{i-1}(x_i) = s''_i(x_i) \qquad i = 2, \dots, n-1$$



One way to impose the conditions in Eq. 86 is through the following relationships:

Eq. 87 $$s''_{i-1}(x_i) = s''_i(x_i) = M_i \qquad i = 2, \dots, n-1$$

A possible way to have these conditions verified is the following one:

Eq. 88 $$s''_i(x) = \frac{M_{i+1}(x - x_i) + M_i(x_{i+1} - x)}{x_{i+1} - x_i} \qquad i = 1, \dots, n-1$$

which gives $s''_i(x_i) = M_i$ and $s''_i(x_{i+1}) = M_{i+1}$ which is exactly Eq. 87. By integration of Eq. 88 we get:

Eq. 89 $$s'_i(x) = \frac{M_{i+1}(x - x_i)^2 - M_i(x_{i+1} - x)^2}{2(x_{i+1} - x_i)} + C_i \qquad i = 1, \dots, n-1$$

With further integration we get:

Eq. 90 $$s_i(x) = \frac{M_{i+1}(x - x_i)^3 + M_i(x_{i+1} - x)^3}{6(x_{i+1} - x_i)} + C_i x + D_i \qquad i = 1, \dots, n-1$$

Considering Eq. 89, the conditions in Eq. 85 give:

Eq. 91 $$\frac{M_i(x_{i+1} - x_{i-1})}{2} = C_i - C_{i-1} \qquad i = 2, \dots, n-1$$

On the other hand, by substituting Eq. 90 in Eq. 84 we get

Eq. 92 $$\frac{M_i(x_i - x_{i-1})^2}{6} + C_{i-1}x_i + D_{i-1} = \frac{M_i(x_{i+1} - x_i)^2}{6} + C_i x_i + D_i = f_i \qquad i = 2, \dots, n-1$$

From these relationships we can write:

$$\begin{cases} C_i x_{i+1} + D_i = f_{i+1} - \dfrac{M_{i+1}(x_{i+1} - x_i)^2}{6} \\ D_i = f_i - \dfrac{M_i(x_{i+1} - x_i)^2}{6} - C_i x_i \end{cases} \Rightarrow \begin{cases} C_i = \dfrac{f_{i+1} - f_i}{x_{i+1} - x_i} - \dfrac{(M_{i+1} - M_i)(x_{i+1} - x_i)}{6} \\ D_i = f_i - \dfrac{M_i(x_{i+1} - x_i)^2}{6} - C_i x_i \end{cases} \Rightarrow$$

Eq. 93 $$\begin{cases} C_i = \frac{f_{i+1}-f_i}{x_{i+1}-x_i} - \frac{(M_{i+1}-M_i)(x_{i+1}-x_i)}{6} \\ D_i = \frac{f_i x_{i+1} - f_{i+1} x_i}{x_{i+1}-x_i} + \frac{M_{i+1}x_i - M_i x_{i+1}}{6}(x_{i+1} - x_i) \end{cases} \qquad i = 1, \ldots, n-1$$

If we substitute Eq. 93 in Eq. 91 we get

$$\frac{M_i(x_{i+1} - x_{i-1})}{2} = \frac{f_{i+1} - f_i}{x_{i+1} - x_i} - \frac{f_i - f_{i-1}}{x_i - x_{i-1}} + \frac{M_i(x_{i+1} - x_{i-1}) - M_{i-1}(x_i - x_{i-1}) - M_{i+1}(x_{i+1} - x_i)}{6} \Rightarrow$$

Eq. 94 $$M_{i-1}\frac{x_i - x_{i-1}}{6} + M_i\frac{x_{i+1} - x_{i-1}}{3} + M_{i+1}\frac{x_{i+1} - x_i}{6} = \frac{f_{i+1} - f_i}{x_{i+1} - x_i} - \frac{f_i - f_{i-1}}{x_i - x_{i-1}} \qquad i = 2, \ldots, n-1$$

These are $n-2$ equations for n unknowns, but we can still put $M_1 = M_n = 0$. By substituting $M_1, M_2, \ldots, M_n$ in Eq. 93 we obtain $C_1, C_2, \ldots, C_{n-1}$ and $D_1, D_2, \ldots, D_{n-1}$. At this point we have the solution from Eq. 90 (9).

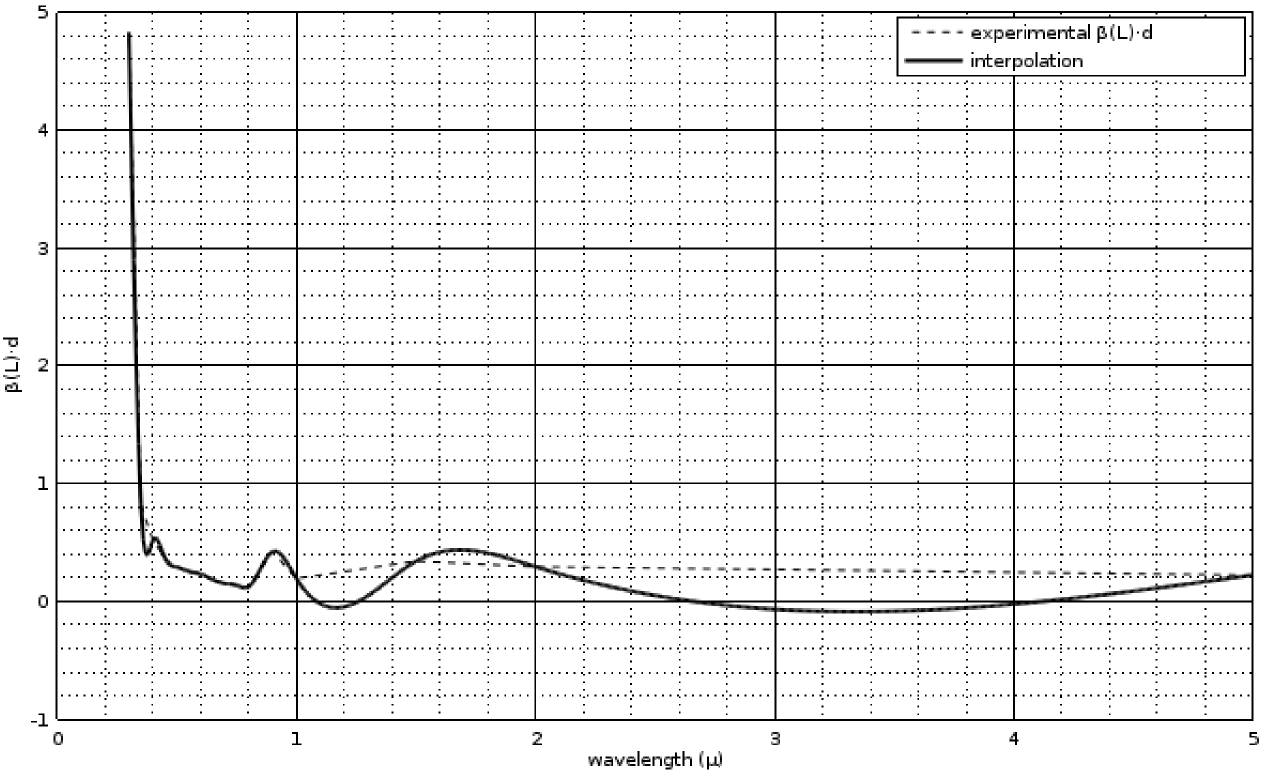


Figure 23. The values of $\beta(L)d$, where $\beta(L)$ is the extinction coefficient in function of the wavelength L and *d* is the thickness of the atmosphere.

3.2.1 Application to the extinction coefficient

If we apply the method of interpolation that we have just introduced to the extinction coefficient in Table 1, we obtain the diagram in Figure 23. The code used to calculate the interpolation is the following one.

CODE 7---

```
% file name = beta_interpolation
% date of creation = 5/11/2019
clear all
n = 16;
% load the array containing beta*d
load beta.mat beta
for i = 1:16
  x(i) = beta (i,2);
  f(i) = beta (i,1);
endfor
% building the matrix of coefficients
matrix (1:n,1:n) = 0.;
matrix (1,1) = 1;
vector (1) = 0.;
for i=2:n-1
  matrix (i,i-1) = ( x(i) - x (i-1) )/6;
  matrix (i,i) = ( x(i+1) - x (i-1) )/3;
  matrix (i,i+1) = ( x(i+1) - x (i) )/6;
  vector (i) = ( f(i+1) - f(i) )/( x(i+1) - x (i) ) - ( f(i) - f(i-1) )/( x(i) - x (i-1) );
endfor
vector (n) = 0.;
matrix (n,n) = 1;
% solving the linear system
M = inverse(matrix)*vector';
% calculating C and D
for i = 1:n-1
  C(i) = ( f(i+1) - f(i) )/( x(i+1) - x (i) ) - ( M(i+1) - M(i) )*( x(i+1) - x (i) )/6;
  D(i) = ( f(i)*x(i+1) - f(i+1)*x(i) )/( x(i+1) - x (i) ) + ( M(i+1)*x(i) - M(i)*x(i+1) )*( x(i+1) - x (i) )/6;
endfor
% the array of the wavelength
N = 471;
L(1) = x(1);
L(N) = x(n);
delta_L = ( L(N) - L(1) )/(N-1);
for j = 2:N-1
  L (j) = L(j-1) + delta_L;
endfor
% the function that interpolates beta*d
S(1) = f(1);
S (N) = f(n);
for j = 2:N-1
  for i=1:n-1
    if ( and(L(j) > x(i), L(j) < x(i+1)) )
      S(j) = C(i)*L(j) + D(i) + ( ( M(i+1)*( L(j) - x(i) )^3 ) + ( M(i)*( x(i+1) - L(j) )^3 ) )/( 6*(x(i+1) - x(i) ) );
    endif
    if ( L(j) == x(i) )
      S(j) = f(i);
    endif
  endfor
endfor
% plotting
plot (beta (1:16, 2), beta (1:16, 1), '--k', "linewidth", 1)
xlabel('wavelength ({\mu})');
ylabel('{\beta}(L){\cdot}d');
```

```
hold on
plot (L (:), S (:), '-k', "linewidth", 2)
legend ('experimental {\beta}(L){\cdot}d', 'interpolation', "location",'NORTHEAST')
grid minor on
save beta_int.mat S
```

----

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